

# Notes on One-loop Calculations in Light-cone Gauge<sup>1</sup>

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## Abstract

Loop calculations in light-cone gauge must confront many technical complexities. We offer here a compendium of detailed light-cone calculations in Yang-Mills theories (with no matter fields). We consistently regulate the  $p^+ = 0$  singularities through discretization of the  $p^+$  component of momentum. Although it is more cumbersome than the Mandelstam-Leibbrandt prescription, this choice has the virtue of employing only positive norm states, retaining manifest unitarity. Some of the results given here are useful for the forthcoming paper [1], specifically the results for the gluon self energy and one-loop vertex corrections.

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# 1 Introduction

In these notes we carry out several one loop calculations using lightcone gauge and employing a novel regularization of Feynman diagrams motivated by the light-cone worldsheet picture of planar diagrams [2, 3]. For its rigorous definition the worldsheet formalism relies on a discretization of  $\sigma, \tau$ , and hence of  $T_i, p_i^+$  the light cone times and longitudinal momenta associated with the various propagators of the diagram. On the other hand, conventional Feynman diagrams require continuous  $T_i, p_i^+$ . The ultraviolet divergences of quantum field theory correspond in lightcone variables to infinities due to integration at large transverse momentum. These transverse momentum infinities will get entangled with, and will spoil, the continuum limit of the  $T_i, p_i^+$  unless they are regulated independently of these longitudinal variables [4–6]. The requirement that this transverse regulator be local on the worldsheet then dictates that it be applied only to the boundary values  $\mathbf{q}_i$  of the worldsheet fields  $\mathbf{q}(\sigma, \tau)$ . Such a cutoff is local in both  $\sigma, \tau$  because it only need be applied at the beginning or end of an internal boundary (because  $\mathbf{q}$  satisfies Dirichlet boundary conditions), i.e. at a point on the worldsheet. It is particularly convenient for our analysis to simply impose a Gaussian cutoff, i.e. to insert in the integrand the factor  $e^{-\delta \sum \mathbf{q}_i^2}$  [5, 7, 8]. This factor can be directly interpreted as a local modification of the worldsheet action.

With  $\delta > 0$  and fixed, the rigorously defined world sheet path integral for each multi-loop planar diagram can be explicitly evaluated on the worldsheet lattice [9] and then the continuum limit of the  $T_i$  for the various propagators can be safely taken. In gauge theories in lightcone gauge it is necessary to keep  $p^+$  discrete until the end, taking the continuum limit only for physical quantities. This is because  $p^+ = 0$  divergences will remain in unphysical intermediate quantities<sup>3</sup>. The result, essentially by construction, reduces to one of the standard representations of the Feynman diagram in momentum space with the regulator factor  $e^{-\delta \sum \mathbf{q}_i^2}$  inserted in the integrand. Because the  $\delta$  regulator is in place and  $p^+$  is discrete, these integrals are manifestly finite. In Section 2 we describe how this reduction takes place.

After this reduction, there remains an almost conventional analysis of the renormalization procedure in the context of this somewhat novel regulator. The novelty stems from the fact that the  $q_i$ ’s, the variables subject to the cut-off, are not the momenta flowing through the propagators. Rather, they are “dual-momentum” variables, one assigned to each loop. There is also a set of external dual-momenta  $q_i^e$ , one assigned to each region between external lines. The momentum flowing through the propagator that separates loop  $i$  from loop  $j$  is the difference  $q_i - q_j$ . Thus the regulator breaks a “translation” symmetry  $q_i^e \rightarrow q_i^e + a$  enjoyed by the *bare* unregulated diagram<sup>4</sup>. Because of this broken symmetry, with  $\delta > 0$  the  $n$ -point function depends on  $n$  dual-momenta rather than on  $n - 1$  actual momenta. Formally the limit  $\delta \rightarrow 0$  should restore the symmetry and the amplitudes should become independent of one of the dual-momenta. Because of ultraviolet divergences, the introduction of counter-terms is necessary to ensure that this happens.

In Section 3 we describe the properties of this regularization in detail. In section 4 we discuss the self-energy and its renormalization through one loop by direct calculation. The one loop three point vertex is calculated in Section 5, and the correct asymptotically free behavior is confirmed. Finally, in section 6 we include some analysis of the box diagrams with maximal and next to maximal helicity violation.

## 2 From Lightcone to the Schwinger Representation

By construction, the evaluation of the worldsheet path integral representing a specific planar Feynman diagram produces a certain discretized version of the usual multi-loop integral. Each propagator appears in its mixed  $\mathbf{p}, p^+ > 0, x^+$  representation

$$\int \frac{dp^-}{2\pi} e^{-ix^+ p^-} \frac{-i}{p^2 + \mu_0^2 - i\epsilon} = \frac{\theta(x^+)}{2p^+} e^{-ix^+ (\mathbf{p}^2 + \mu_0^2)/2p^+}$$

<sup>3</sup>Another approach is the Mandelstam-Leibbrandt principal value prescription, which retains continuous  $p^+$  but gives up manifest unitarity. We prefer retaining unitarity.

<sup>4</sup>Because the regulator only cuts off the transverse components of  $\mathbf{q}$ , the translation symmetry in the longitudinal momenta remains unbroken.

$$\rightarrow \frac{\theta(\tau)}{2p^+} e^{-\tau(\mathbf{p}^2 + \mu_0^2)/2p^+}. \quad (1)$$

The Feynman integration is over all independent  $\tau_i, p_i^+, \mathbf{p}_i$ . However the worldsheet lattice formalism produces instead sums over discretized  $\tau_i = k_i a, p_i^+ = l_i m$ , while keeping the  $\mathbf{p}_i$  integrals continuous. However, in the presence of the regulator  $\delta > 0$  described in the introduction, one can safely replace all of the discretized sums by continuous integrals.

We would like to now show that, for cubic scalar vertices, these perhaps unfamiliar lightcone multi-loop integrals are identical to the covariant Feynman integrals in which each propagator is written in a Schwinger representation.

$$\frac{1}{p^2 + \mu^2} = \int_0^\infty dT e^{-T(p^2 + \mu^2)}. \quad (2)$$

Indeed, it is straightforward to show that the number of independent  $\tau_i, p_i^+$  in the diagram's lightcone representation is precisely equal to the number of  $T_i$  in the diagram's Schwinger representation. If one explicitly carries out the Gaussian integrals in the two representations by completing the square the remaining integrals in the two representations will be of the same dimensionality. The integrands are very similar except that the determinant prefactor from the lightcone is raised to the  $(D-2)/2$  power compared to the  $D/2$  power in the Schwinger representation. One can make the exponentials in the integrands identical by changing integration variables from the  $\tau_i, p_i^+$  to appropriate  $T_i$ . It then turns out that the Jacobian for this change of variables supplies the missing determinant factors.

To understand why this happens, just consider the transform to light-cone representation of the Schwinger representation:

$$\begin{aligned} -i \int \frac{dp^-}{2\pi} e^{-ix^+ p^-} \int idT e^{-iT(\mathbf{p}^2 + \mu_0^2 - 2p^+ p^- - i\epsilon)} &= -i \int idT \delta(x^+ - 2p^+ T) e^{-iT(\mathbf{p}^2 + \mu_0^2)} \\ &\rightarrow \int dT \delta(\tau - 2p^+ T) e^{-T(\mathbf{p}^2 + \mu_0^2)}. \end{aligned} \quad (3)$$

From this result we see that the appropriate change of variables is  $T = \tau/2p^+$ . It is interesting and satisfying that the passage to imaginary  $x^+$  in the lightcone representation is completely equivalent to writing the Schwinger representation with a real exponential, which of course is only meaningful after the Wick rotation to Euclidean space.

For the rest of the discussion of renormalization we need no longer refer to the explicit worldsheet representation. We only have to write the usual covariant rules using dual momenta  $q_i$ , and insert the regulator factor  $e^{-\delta \sum_i q_i^2}$ . Once we have established the form of the counter-terms required for renormalization we shall return to give their worldsheet representation at the end of the article.

### 3 Regularization

Draw an arbitrary planar diagram so that its lines divide the plane into different regions, the external lines all going off to infinity. Then the external lines bound infinite regions, and the finite regions fill each loop of the multi-loop diagram. For each loop introduce a momentum  $q_i^\mu$ , assigned to the loop's region. Then each propagator line separates two regions, say  $i_1$  and  $i_2$ , and the propagator's momentum is then taken to be  $q_{i_1} - q_{i_2}$ , and momentum is automatically conserved. We regulate each diagram by including in the integrand the factor  $e^{-\delta \sum_{i=1}^L q_i^2}$ . Since we are using a light-cone world sheet we only cut off the transverse momentum integrals, because we want to preserve longitudinal boost invariance<sup>5</sup>. This regularization sacrifices full Lorentz invariance, but respects the  $O(D-2)$  rotational invariance in transverse space as well as the longitudinal boost invariance. The transverse boost invariances generated by  $M^{\pm i}$  are broken, and it will require counter-terms to restore them in the renormalizable case.

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<sup>5</sup>One could easily extent the cutoff to the longitudinal variables, but then the light-cone interpretation would be obscured.

Without loss of generality, we can and do restrict attention to proper (*i.e.* connected one particle irreducible) diagrams, with propagators removed from external legs. Such diagrams never have tadpole sub-diagrams, which would be problematic for the lightcone description (because  $p^+ > 0$ ), though not for a covariant description. The only 1PIR diagram involving a tadpole is the one-point function itself,  $\langle \Phi \rangle$ . It is true that the lightcone description has no convenient representation of the one point function. However, in a covariant description, the only effect of tadpoles as sub-diagrams in larger (improper) diagrams is pure mass renormalization, which means their effect can be absorbed in an additive constant in the self-energy counter-term. In this article we assume that this is always done and therefore drop tadpoles completely. Then we can freely pass back and forth between light-cone and covariant descriptions, as long as we refrain from considering the one-point function itself. Since the one-point function cannot be directly measured in any case, this is no limitation on the lightcone description. If needed, the value of the one-point function can be related via the field equations to  $\langle \Phi^2 \rangle$ , which in turn can be extracted from the high momentum limit of the two point function.

It is convenient to employ the Schwinger representation of each propagator (2):

$$\frac{1}{p^2 + \mu^2} = \int_0^\infty dT e^{-T(p^2 + \mu^2)}$$

which enables the execution of all loop momentum integrals by completing the square in the exponents of the Gaussian integrals. To describe this for an  $L$  loop diagram, assemble the loop momenta in an  $L$  dimensional vector  $q$  and call  $M_0$  the  $L \times L$  symmetric matrix that describes the quadratic terms in the  $q_i$ , so the exponent reads

$$-q^T \cdot (M_0 + \delta)q + v^T \cdot q + q^T \cdot v - B \quad (4)$$

where the  $L$ -vector  $v$  describes the couplings to the momenta assigned to the external regions and  $B$  is a bilinear form in those external momenta. It is understood that  $\delta \neq 0$  only for the transverse components. Then the result of the loop integrations is just

$$\begin{aligned} & \frac{\pi^L}{\det M_0} \left( \frac{\pi^L}{\det(\delta + M_0)} \right)^{(D-2)/2} \exp \left\{ -B + v^T \cdot \frac{1}{\delta + M_0} v \right\} = \\ & \frac{\pi^L}{\det M_0} \left( \frac{\pi^L}{\det(\delta + M_0)} \right)^{(D-2)/2} \exp \left\{ -B + v^T \cdot \frac{1}{M_0} v - \delta v^T \cdot \frac{1}{M_0} \cdot \frac{1}{\delta + M_0} v \right\} \end{aligned} \quad (5)$$

We see that the shift of  $M_0$  by  $\delta$  regulates the integration region near the zeroes of the determinant, which is the source of ultraviolet divergences in the diagram. The first two terms in the exponent are manifestly Lorentz invariant and are precisely what they would be in the unregulated theory. The last term in the exponent breaks Lorentz invariance because it depends explicitly on the transverse momentum components. If we could argue that it were negligible (as  $\delta \rightarrow 0$ ), we could assert from the known proofs of renormalizability that all divergences as  $\delta \rightarrow 0$  could be covariantly absorbed in the renormalization of mass  $\mu$  and coupling  $g$  to all orders in perturbation theory.

The term in question is nominally of order  $O(\delta)$  but since it also depends on the  $T_i$ 's we must check this estimate more carefully. First note that  $q_0 \equiv (\delta + M_0)^{-1} v$  is in fact the location of the minimum of a bilinear form in the  $q_i$ 's that has the interpretation as the potential energy of  $L$  particles tied to each other and to the fixed external momenta with a bunch of springs with spring constants  $T_i > 0$  and to the origin with springs of spring constant  $\delta$ . It is obvious that the resulting equilibrium has every  $q_{0i}$  within the simplex with vertices at the origin and the external momenta. If  $\delta = 0$  they are within the simplex with vertices at the external momenta. In either case it follows that  $|q_{0i}|$  is uniformly bounded by the largest external momentum. Thus we can conclude that the term in question is uniformly bounded over the whole integration region by  $L\delta|q_{ext}|_{max}^2$ . Thus the  $O(\delta)$  estimate is rigorous.

Even so, Lorentz non-covariance can survive due to ultraviolet divergences of degree  $1/\delta$  or worse which can overwhelm the  $O(\delta)$  suppression. Fortunately, in a renormalizable theory we can isolate where these

divergences can occur and accordingly identify the subtractions necessary to remove these contributions which would violate Lorentz invariance. Indeed the ultraviolet divergences in vertex parts are superficially linear in momentum ( $1/\sqrt{\delta}$ ) while those in self-energy parts are quadratic in momentum ( $O(1/\delta)$ ). Thus the Lorentz violations due to the term in the exponent will be associated with self-energy divergences, but of course we must follow their impact in sub-diagrams of larger diagrams as well. That term will be negligible in three and higher point diagrams, but there are also some  $\delta$  artifacts due to the linear momentum factors in the cubic vertex, which survive because of the latter's superficial linear divergence. Thus we should expect the associated counter-terms to involve at most three factors of the gauge field.

## 4 Gluon Self Energy

In order to acquaint the reader with some of the novelties of calculations using the  $\delta$  regulator, we carry out in this section a direct calculation of the self energy through one loop, with an explicit separation of all divergences and Lorentz-violating artifacts. We call the bare gluon self-energy  $\Pi_0^{ij}$ , but it is convenient to calculate  $Z\Pi_0^{ij}$  and absorb the factor of  $Z$  in the bare coupling by defining the renormalized coupling  $g = g_0 Z^{3/2}/Z_1$ , where  $Z_1$  is the three vertex renormalization constant. In other words we write down the Feynman rules in terms of renormalized mass and coupling, canceling infinities against the self-energy counter-term  $Z\Pi_{\text{C.T.}}^{ij}$  and the three vertex counter-term  $g(Z_1 - 1)A^3$ , which are included in the Feynman rules, rather than absorbing them in redefinitions of the bare parameters.

Choose the complex basis for the gluon polarization  $1, 2: \wedge = (1 + i2)/\sqrt{2}, \vee = (1 - i2)/\sqrt{2}$ . The unsubtracted one-loop self-energy diagrams have the values

$$\begin{aligned} Z\Pi_0^{\wedge\vee} &= Z\Pi_0^{\vee\wedge} = \frac{g^2 N_c}{4\pi^2} \int_0^\infty \frac{dT_1 dT_2}{T_1 + T_2} \left[ \frac{1}{(T_1 + T_2 + \delta)^2} + \frac{\delta^2 [T_1 \mathbf{q} + T_2 \mathbf{q}']^2}{(T_1 + T_2)^2 (T_1 + T_2 + \delta)^3} \right] \\ &\quad \left( 1 + \frac{(T_1 + T_2)^2}{T_1^2} + \frac{(T_1 + T_2)^2}{T_2^2} \right) \exp \left\{ -\frac{T_1 T_2}{T_1 + T_2} (q - q')^2 - \delta \frac{(T_1 \mathbf{q} + T_2 \mathbf{q}')^2}{(T_1 + T_2)(T_1 + T_2 + \delta)} \right\} \\ &= \frac{g^2 N_c}{4\pi^2} \int_0^\infty dT \int_0^1 dx \left[ \frac{1}{(T + \delta)^2} + \frac{\delta^2 [x\mathbf{q} + (1-x)\mathbf{q}']^2}{(T + \delta)^3} \right] \\ &\quad \left( 1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \right) \exp \left\{ -Tx(1-x)(q - q')^2 - \frac{\delta T}{T + \delta} (x\mathbf{q} + (1-x)\mathbf{q}')^2 \right\} \end{aligned} \quad (6)$$

$$\begin{aligned} Z\Pi_0^{\vee\vee} &= \frac{g^2 N_c}{2\pi^2} \int_0^\infty dT \int_0^1 dx \frac{\delta^2 [xq^\vee + (1-x)q'^\vee]^2}{(T + \delta)^3} \\ &\quad \exp \left\{ -Tx(1-x)(q - q')^2 - \frac{\delta T}{T + \delta} (x\mathbf{q} + (1-x)\mathbf{q}')^2 \right\} \rightarrow \frac{g^2 N_c}{4\pi^2} \int_0^1 dx [xq^\vee + (1-x)q'^\vee]^2 \end{aligned} \quad (7)$$

And  $\Pi_0^{\wedge\wedge}$  is obtained from  $\Pi_0^{\vee\vee}$  by the substitution  $\vee \rightarrow \wedge$ . Note that these two quantities are simply quadratic polynomials in  $\mathbf{q}, \mathbf{q}'$ , so a counter-term can be introduced to cancel them completely. The quadratic divergence in  $\Pi_0^{\wedge\vee}$  can be simply extracted with an integration by parts. We observe that

$$\begin{aligned} \left[ \frac{1}{(T + \delta)^2} + \frac{\delta^2 [x\mathbf{q} + (1-x)\mathbf{q}']^2}{(T + \delta)^3} \right] \exp \left\{ -\frac{\delta T}{T + \delta} (x\mathbf{q} + (1-x)\mathbf{q}')^2 \right\} &= \\ -\frac{\partial}{\partial T} \frac{1}{T + \delta} \exp \left\{ -\frac{\delta T}{T + \delta} (x\mathbf{q} + (1-x)\mathbf{q}')^2 \right\} \end{aligned} \quad (8)$$

So we can rewrite the self energy as

$$\begin{aligned} Z\Pi_0^{\wedge\vee} &= -\frac{g^2 N_c}{4\pi^2} (q - q')^2 \int_0^1 dx \left( x(1-x) + \frac{1-x}{x} + \frac{x}{1-x} \right) I(H\delta) \\ &\quad + \frac{g^2}{4\pi^2} \frac{1}{\delta} \int_0^1 dx \left( 1 + \frac{1}{x^2} + \frac{1}{(1-x)^2} \right) \end{aligned} \quad (9)$$

$$H \equiv x(1-x)(q-q')^2 \quad (10)$$

$$I(H\delta) \equiv \int_0^\infty du \frac{e^{-uH\delta - u\delta(x\mathbf{q} + (1-x)\mathbf{q}')^2/(1+u)}}{1+u} \quad \xrightarrow{\delta \rightarrow 0} -\gamma - \ln(H\delta) = -\ln(H\delta e^\gamma) \quad (11)$$

where  $\gamma = -\Gamma'(1)/\Gamma(1)$  is Euler's constant. Clearly the  $x$  integrals diverge at the end points of integration. These divergences are spurious artifacts of the light-cone gauge and have nothing to do with the usual ultraviolet divergences of the gauge theory. They must cancel in physical quantities without invoking renormalization or counter-terms. In our approach the  $x$  integration just corresponds to integration over the location on the worldsheet of the boundary representing the loop. On the world sheet lattice this location is an integer  $l$  with  $x = l/M$  and  $M$  is the discretized total plus momentum entering the self energy:  $q^+ = mM$ . The discreteness of  $p^+$  regulates the endpoint  $x$  divergences.

Let us discuss first the fate of the quadratic  $1/\delta$  divergence, which for discrete  $p^+$  reads:

$$\frac{g^2}{4\pi^2} \frac{1}{M\delta} \sum_{l=1}^{M-1} \left( 1 + \frac{M^2}{l^2} + \frac{M^2}{(M-l)^2} \right) \sim \frac{g^2}{4\pi^2} \frac{1}{\delta} \left( \frac{\pi^2}{3} M - 1 + O\left(\frac{1}{M}\right) \right) \quad (12)$$

where the right side indicates the large  $M$  behavior of the sums. The term linear in  $M = p^+/m$  cannot be canceled by a gluon self mass, because it is linear in  $p^+$ . However, precisely because it is linear in  $p^+$ , it represents a constant  $-g^2 M/(24p^+\delta) = -g^2/(24m\delta)$  added to the energy  $p^-$  of each gluon. A constant added to  $p^-$  can be interpreted as energy associated with the boundary of the worldsheet representing that gluon, in other words with a boundary cosmological constant. If we start with a nonzero boundary cosmological constant  $\lambda_b$  in zeroth order, we can tune its value to cancel the linear terms in  $p^+$  generated by loop effects. Its lowest order value is then

$$\lambda_b = +\frac{g^2}{24m\delta} \quad (13)$$

After this cancellation, there is left behind a constant which can be canceled by a gluon mass counter-term  $\delta\mu^2$ . So to this order

$$\delta\mu^2 = \frac{g^2}{4\pi^2\delta} \quad (14)$$

Of course, the gluon mass is zero in tree approximation, but since loop corrections generate a gluon mass, the tree value must be non-zero and adjusted to cancel the loop contributions order by order in perturbation theory. A nonzero mass at tree level violates gauge invariance, which means a violation of Lorentz invariance in the completely fixed lightcone gauge. So an alternative prescription is: in lightcone gauge, allow a nonzero gluon mass  $\mu_0^2$  as an input parameter, and calculate physical quantities as functions of this parameter. Finally, choose a value of this parameter that restores Lorentz invariance. Note that to one loop,  $\mu^2 = 0$  requires a tachyonic gluon mass:  $\mu_0^2 = -\delta\mu^2$ .

Next we turn to the logarithmic divergences in the self energy. For dimensional reasons write  $H\delta = (H/\mu^2)(\mu^2\delta)$ , so as  $\delta \rightarrow 0$   $I(H\delta) \rightarrow \Gamma'(1) - \ln(H/\mu^2) - \ln(\mu^2\delta)$ . Also call  $p = q' - q$  and remember that  $x = q^+/p^+$ . Including the above mentioned counter-terms, we then find

$$Z\Pi^{\wedge\vee} - 2p^+\lambda_b + \delta\mu^2 = \frac{g^2 N_c}{4\pi^2} p^2 \sum_{q^+} \left( x(1-x) - 2 + \left[ \frac{1}{q^+} + \frac{1}{p^+ - q^+} \right] \right) \ln \{ x(1-x)p^2\delta e^\gamma \} \quad (15)$$

$$\sim \frac{g^2 N_c}{4\pi^2} p^2 \left( -\frac{11}{6} + \sum_{q^+} \left[ \frac{1}{q^+} + \frac{1}{p^+ - q^+} \right] \right) \ln(\mu^2\delta) + \text{Finite} \quad (16)$$

where Finite means with respect to  $\delta \rightarrow 0$ . We may associate this log divergent contribution with a  $p^+$  dependent wave function renormalization factor

$$Z_{p^+} = 1 + \frac{g^2 N_c}{4\pi^2} \left( -\frac{11}{6} + \sum_{q^+} \left[ \frac{1}{q^+} + \frac{1}{p^+ - q^+} \right] \right) \ln(\mu^2\delta) \quad (17)$$

Note that this  $Z_{p^+} < 1$  by virtue of the (divergent)  $q^+$  sums, in accordance with the requirements of unitarity. Of course, we really want renormalization constants to be independent of the momenta, so we define instead the wave function renormalization

$$\begin{aligned} Z_3 &= 1 + \frac{g^2 N_c}{4\pi^2} \int_0^1 dx (x(1-x) - 2) \ln \{x(1-x)\mu^2 \delta e^\gamma\} \\ &= 1 + \frac{g^2 N_c}{4\pi^2} \left\{ -\frac{11}{6} \ln \{\mu^2 \delta e^\gamma\} + \frac{67}{18} \right\} \end{aligned} \quad (18)$$

which is *larger* than 1. We then leave the  $p^+$  dependent part of the log divergences in the definition of the “renormalized”  $\Pi$  and must find that these are precisely canceled by other contributions. Indeed we will find these canceling contributions in the two vertex renormalizations each (internal) propagator attaches to. Naturally, the cancellation is incomplete on the external lines. Removing a factor of  $Z_3$  from the gluon propagator corresponds to using the “renormalized” self energy

$$Z\Pi_R^{\wedge\vee} \equiv \frac{g^2 N_c}{4\pi^2} p^2 \left\{ \sum_{q^+} \left[ \frac{1}{q^+} + \frac{1}{p^+ - q^+} \right] \ln \{x(1-x)p^2 \delta e^\gamma\} - \frac{11}{6} \ln \frac{p^2}{\mu^2} \right\} \quad (19)$$

where we must find that the  $\delta$  dependence cancels in physical quantities. To simplify future equations, we give the anomalous quantity in braces a name:

$$\mathcal{A}(p^2, p^+) \equiv \sum_{q^+} \left[ \frac{1}{q^+} + \frac{1}{p^+ - q^+} \right] \ln \{x(1-x)p^2 \delta e^\gamma\}. \quad (20)$$

We shall find this quantity occurring in vertex calculations. Then

$$Z\Pi_R^{\wedge\vee} \equiv \frac{g^2 N_c}{4\pi^2} p^2 \left\{ \mathcal{A}(p^2, p^+) - \frac{11}{6} \ln \frac{p^2}{\mu^2} \right\} \quad (21)$$

## 5 Three Point Function and one-loop coupling renormalization

### 5.1 Triangle graph

The tree vertex is

$$2g \frac{p_3^+}{p_1^+ p_2^+} \mathbf{K}_{12} \equiv 2g \frac{p_3^+}{p_1^+ p_2^+} (p_1^+ \mathbf{p}_2 - p_2^+ \mathbf{p}_1) \quad (22)$$

In this section we turn to the 1PIR loop corrections to the cubic vertex. First consider the maximal helicity violating triangle, which must be finite because there is no tree contribution to this amplitude:

$$\begin{aligned} \Gamma^{\wedge\wedge\wedge} &= -\frac{g^3}{\pi^2} \int_0^\infty \frac{dT_1 dT_2 dT_3}{T_{13}(T_{13} + \delta)} \exp \left\{ -\frac{\delta(\mathbf{k}_0 T_3 + \mathbf{k}_1 T_1 + \mathbf{k}_2 T_2)^2}{T_{13}(T_{13} + \delta)} \right\} \\ &\quad \exp \left\{ -\frac{T_1 T_3 (k_1 - k_0)^2 + T_1 T_2 (k_1 - k_2)^2 + T_2 T_3 (k_2 - k_0)^2}{T_1 + T_2 + T_3} \right\} \\ &\quad \prod_{j=1}^3 \left\{ \frac{T_j K^\wedge}{p_{j-1}^+ T_{13}} - \frac{\delta(T_1 k_1^\wedge + T_2 k_2^\wedge + T_3 k_0^\wedge)}{T_{13}(\delta + T_{13})} \right\} \end{aligned} \quad (23)$$

$$\mathbf{K} = \mathbf{K}_{12} = p_3^+ \mathbf{k}_1 + p_1^+ \mathbf{k}_2 + p_2^+ \mathbf{k}_0 = p_1^+ (\mathbf{k}_2 - \mathbf{k}_1) - p_2^+ (\mathbf{k}_1 - \mathbf{k}_0). \quad (24)$$

We see by inspection that the  $\delta \rightarrow 0$  limit of this amplitude is perfectly finite.

$$\begin{aligned} \Gamma^{\wedge\wedge\wedge} &\sim -\frac{g^3}{\pi^2} \int_0^\infty \frac{dT_1 dT_2 dT_3}{p_1^+ p_2^+ p_3^+ T_{13}^5} T_1 T_2 T_3 (K^\wedge)^3 \exp \left\{ -\frac{T_1 T_3 (k_1 - k_0)^2 + T_1 T_2 (k_1 - k_2)^2 + T_2 T_3 (k_2 - k_0)^2}{T_1 + T_2 + T_3} \right\} \\ &\sim -\frac{g^3}{\pi^2} \frac{(K^\wedge)^3}{p_1^+ p_2^+ p_3^+} \int_{x+y < 1} dx dy \frac{xy(1-x-y)}{x(1-x-y)(k_1 - k_0)^2 + xy(k_1 - k_2)^2 + y(1-x-y)(k_2 - k_0)^2} \end{aligned} \quad (25)$$

The other maximal helicity violating amplitude  $\Gamma^{\vee\vee\vee}$  is obtained from this result by the substitution  $K^\wedge \rightarrow K^\vee$ . For the contribution of the triangle to on-shell scattering at one loop, two of the  $p_i^2 = 0$ , and the amplitude simplifies to

$$\Gamma^{\wedge\wedge\wedge} \sim -\frac{g^3}{6\pi^2} \frac{(K^\wedge)^3}{p_1^+ p_2^+ p_3^+ p^2} \quad (26)$$

where  $p$  is the momentum of the off-shell leg.

The amplitudes,  $\Gamma^{\wedge\wedge\vee}$ ,  $\Gamma^{\vee\vee\wedge}$ , and those obtained by cyclic permutation, are more challenging because they contain both infra-red and ultraviolet divergences. We shall work out the first case, together with its cyclic permutations, in complete detail.. The amplitude for  $\Gamma^{\wedge\wedge\vee}$ , after the shift of loop momentum  $\mathbf{q}$  which completes the square in the exponent, is

$$\begin{aligned} \Gamma^{\wedge\wedge\vee} = & -4g^3 \sum_{q^+} \int dT_1 dT_2 dT_3 \delta(T_{13}q^+ - T_{12}p_1^+ - T_2p_2^+) \int \frac{d^2q}{(2\pi)^3} \exp \left\{ -\frac{\delta(\mathbf{k}_0 T_3 + \mathbf{k}_1 T_1 + \mathbf{k}_2 T_2)^2}{T_{13}(T_{13} + \delta)} \right\} \\ & \exp \left\{ -(T_{13} + \delta) \mathbf{q}^2 - \frac{T_1 T_3 (k_1 - k_0)^2 + T_1 T_2 (k_1 - k_2)^2 + T_2 T_3 (k_2 - k_0)^2}{T_1 + T_2 + T_3} \right\} \\ & \left\{ A_3^{\wedge\wedge\vee} (q + K_1)^\wedge (q + K_2)^\wedge (q + K_3)^\vee + A_2^{\wedge\wedge\vee} (q + K_1)^\wedge (q + K_2)^\vee (q + K_3)^\wedge \right. \\ & \left. + A_1^{\wedge\wedge\vee} (q + K_1)^\vee (q + K_2)^\wedge (q + K_3)^\wedge \right\} \end{aligned} \quad (27)$$

where for brevity we have defined<sup>6</sup>

$$\begin{aligned} A_1^{\wedge\wedge\vee} &= \frac{p_1^{+2} p_3^{+2}}{(q^+ - p_1^+)^2 q^{+2}}, & A_2^{\wedge\wedge\vee} &= \frac{p_2^{+2} p_3^{+2}}{(q^+ - p_1^+)^2 (p_1^+ + p_2^+ - q^+)^2} \\ A_3^{\wedge\wedge\vee} &= \frac{q^{+2}}{(p_1^+ + p_2^+ - q^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{q^{+2}} \end{aligned} \quad (30)$$

By  $p^+$  conservation at least one of the  $p_i^+$  is positive and at least one is negative. For definiteness we choose  $p_1^+ > 0$  and  $p_3^+ < 0$ . Then  $p_2^+$  could have either sign. We shall work out in detail the case  $p_2^+ > 0$ . We can then obtain the results for the case  $p_2^+ < 0$  by the following argument. Consider the expression for the amplitude with  $p_1 \rightarrow p'_1 = -p_3$ ,  $p_2 \rightarrow p'_2 = -p_2$ ,  $p_3 \rightarrow p'_3 = -p_1$ ,  $\mathbf{k}_0 \rightarrow \mathbf{k}'_0 = \mathbf{k}_0$ ,  $\mathbf{k}_1 \rightarrow \mathbf{k}'_1 = \mathbf{k}_2$ ,  $\mathbf{k}_2 \rightarrow \mathbf{k}'_2 = \mathbf{k}_1$ . For clarity we also identify the new Schwinger parameters as  $T'_1 = T_2$ ,  $T'_2 = T_1$ , and  $T'_3 = T_3$ . Then, by inspection we find that

$$\Gamma^{\wedge\wedge\vee}(p_i, \mathbf{k}_i) = \Gamma^{\vee\wedge\wedge}(p'_i, \mathbf{k}'_i), \quad \Gamma^{\vee\wedge\wedge}(p_i, \mathbf{k}_i) = \Gamma^{\wedge\wedge\vee}(p'_i, \mathbf{k}'_i), \quad \Gamma^{\wedge\vee\wedge}(p_i, \mathbf{k}_i) = \Gamma^{\vee\wedge\wedge}(p'_i, \mathbf{k}'_i) \quad (31)$$

We see that  $p_1^+ > 0, p_2^+ < 0, p_3^+ < 0$  implies  $p_1'^+ > 0, p_2'^+ > 0, p_3'^+ < 0$ , so we can read off the answer for  $p_2^+ < 0$  from the result for  $p_2'^+ > 0$ .

<sup>6</sup>The amplitudes for the other spin configurations,  $\Gamma^{\vee\vee\wedge}$  and  $\Gamma^{\wedge\vee\wedge}$ , are obtained by modifying the  $A_i^{\wedge\wedge\vee}$  appropriately:

$$\begin{aligned} A_1^{\wedge\wedge\vee} &\rightarrow A_1^{\vee\wedge\wedge} = \frac{q^{+2}}{(p_1^+ - q^+)^2} + \frac{(p_1^+ - q^+)^2}{q^{+2}} & A_2 &\rightarrow A_2^{\vee\wedge\wedge} = \frac{p_2^{+2} p_1^{+2}}{(q^+ - p_1^+)^2 (p_1^+ + p_2^+ - q^+)^2} \\ A_3^{\wedge\wedge\vee} &\rightarrow A_3^{\vee\wedge\wedge} = \frac{p_3^{+2} p_1^{+2}}{q^{+2} (p_1^+ + p_2^+ - q^+)^2} \end{aligned} \quad (28)$$

$$\begin{aligned} A_1^{\wedge\wedge\vee} &\rightarrow A_1^{\wedge\vee\wedge} = \frac{p_1^{+2} p_2^{+2}}{q^{+2} (p_1^+ - q^+)^2} & A_2^{\wedge\wedge\vee} &\rightarrow A_2^{\wedge\vee\wedge} = \frac{(q^+ - p_1^+)^2}{(p_1^+ + p_2^+ - q^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{(q^+ - p_1^+)^2} \\ A_3^{\wedge\wedge\vee} &\rightarrow A_3^{\wedge\vee\wedge} = \frac{p_3^{+2} p_2^{+2}}{q^{+2} (p_1^+ + p_2^+ - q^+)^2} \end{aligned} \quad (29)$$



In view of the divergences at  $q^+ = 0, p_1^+, p_1^+ + p_2^+$ , we have kept the integral over  $q^+$  in discretized form, where setting  $q^+ = lm$  and  $p_i^+ = M_i m$ ,  $\sum_{q^+}$  means  $m \sum_{l=1, l \neq M_1}^{M_1+M_2-1}$ . Also the  $K_i$  are given by

$$\mathbf{K}_1 = \frac{T_2}{T_{13}p_1^+} \mathbf{K} - \frac{\delta(T_1 \mathbf{k}_1 + T_2 \mathbf{k}_2 + T_3 \mathbf{k}_0)}{T_{13}(\delta + T_{13})} \quad (32)$$

$$\mathbf{K}_2 = \frac{T_3}{T_{13}p_2^+} \mathbf{K} - \frac{\delta(T_1 \mathbf{k}_1 + T_2 \mathbf{k}_2 + T_3 \mathbf{k}_0)}{T_{13}(\delta + T_{13})} \quad (33)$$

$$\mathbf{K}_3 = \frac{T_1}{T_{13}p_3^+} \mathbf{K} - \frac{\delta(T_1 \mathbf{k}_1 + T_2 \mathbf{k}_2 + T_3 \mathbf{k}_0)}{T_{13}(\delta + T_{13})} \quad (34)$$

We have for the  $\mathbf{q}$  integration

$$\int d^2 q e^{-(T_{13}+\delta)\mathbf{q}^2} = \frac{\pi}{T_{13}+\delta}, \quad \int d^2 q q^\wedge q^\vee e^{-(T_{13}+\delta)\mathbf{q}^2} = \frac{\pi}{2(T_{13}+\delta)^2} \quad (35)$$

$$\int d^2 q q^\wedge e^{-(T_{13}+\delta)\mathbf{q}^2} = \int d^2 q q^\vee e^{-(T_{13}+\delta)\mathbf{q}^2} = \int d^2 q q^\wedge q^\wedge e^{-(T_{13}+\delta)\mathbf{q}^2} = \int d^2 q q^\wedge q^\wedge q^\vee e^{-(T_{13}+\delta)\mathbf{q}^2} = 0 \quad (36)$$

From which we obtain

$$\begin{aligned} \Gamma^{\wedge\wedge\vee} &= -\frac{g^3}{4\pi^2} \sum_{q^+} \int dT_1 dT_2 dT_3 \delta(T_{13}q^+ - T_{12}p_1^+ - T_2p_2^+) \\ &\quad \exp \left\{ -\frac{T_1 T_3 (k_1 - k_0)^2 + T_1 T_2 (k_1 - k_2)^2 + T_2 T_3 (k_2 - k_0)^2}{T_1 + T_2 + T_3} - \frac{\delta(\mathbf{k}_0 T_3 + \mathbf{k}_1 T_1 + \mathbf{k}_2 T_2)^2}{T_{13}(T_{13} + \delta)} \right\} \\ &\quad \left\{ \left[ \frac{2K_1^\wedge K_2^\wedge K_3^\vee}{T_{13} + \delta} + \frac{K_1^\wedge + K_2^\wedge}{(T_{13} + \delta)^2} \right] A_3 + \left[ \frac{2K_1^\wedge K_2^\vee K_3^\wedge}{T_{13} + \delta} + \frac{K_1^\wedge + K_3^\wedge}{(T_{13} + \delta)^2} \right] A_2 \right. \\ &\quad \left. + \left[ \frac{2K_1^\vee K_2^\wedge K_3^\wedge}{T_{13} + \delta} + \frac{K_2^\wedge + K_3^\wedge}{(T_{13} + \delta)^2} \right] A_1 \right\} \end{aligned} \quad (37)$$

Now consider the  $\delta \rightarrow 0$  limit. The last term in the exponent is uniformly  $O(\delta)$  and can be dropped since the divergences at small  $T_k$  are at worst logarithmic. The second terms in the expressions for the  $K_i$  are  $O(\delta)$  for finite  $T_i$ , but  $O(1)$  when all the  $T_i$  are  $O(\delta)$ . That region of integration is negligible for the  $K_1 K_2 K_3$  term, but not for the terms linear in the  $K_i$ . Thus it is legitimate to make the substitutions

$$2K_1^\wedge K_2^\vee K_3^\wedge \rightarrow \frac{T_1 T_2 T_3}{T_{13}^3 p_1^+ p_2^+ p_3^+} 2K^\wedge K^\vee K^\wedge = \frac{T_1 T_2 T_3}{T_{13}^3 p_1^+ p_2^+ p_3^+} \mathbf{K}^2 K^\wedge \quad (38)$$

$$K_i + K_{i+1} \rightarrow \frac{T_{i+1}p_{i+1}^+ + T_{i+2}p_i^+}{p_i^+ p_{i+1}^+ T_{13}} K^\wedge - 2\delta \frac{(T_1 \mathbf{k}_1 + T_2 \mathbf{k}_2 + T_3 \mathbf{k}_0)}{T_{13}(\delta + T_{13})} \quad (39)$$

The contribution of the second term on the right of (39) to the integration comes solely from the region of all  $T_i = O(\delta)$  so the exponential factor can be replaced by unity and the integrals to be done can be simplified to

$$I^i = \int dT_1 dT_2 dT_3 \delta(T_{13}q^+ - T_{12}p_1^+ - T_2p_2^+) \frac{\delta T_i}{T_{13}(\delta + T_{13})^3} \quad (40)$$

It is most convenient to use the delta function to eliminate  $T_3$  in favor of  $T_{1,2}$  when  $q^+ < p_1^+$  but  $T_2$  in favor of  $T_{1,3}$  when  $q^+ > p_1^+$ . Then we find

$$(I_{<}^1, I_{<}^2) = \frac{q^{+2}}{4p_1^{+2}(p_1^+ + p_2^+)^2} (p_1^+ + p_2^+, p_1^+), \quad I_{<}^3 = \frac{q^+}{4p_1^+(p_1^+ + p_2^+)} \left[ 2 - \frac{q^+}{p_1^+} - \frac{q^+}{p_1^+ + p_2^+} \right] \quad (41)$$

$$(I_{>}^1, I_{>}^3) = \frac{(p_1^+ + p_2^+ - q^+)^2}{4p_2^{+2}(p_1^+ + p_2^+)^2} (p_1^+ + p_2^+, p_2^+), \quad I_{>}^2 = \frac{p_1^+ + p_2^+ - q^+}{4p_2^+(p_1^+ + p_2^+)} \left[ \frac{q^+ - p_1^+}{p_2^+} + \frac{q^+}{p_1^+ + p_2^+} \right] \quad (42)$$

Note that the second line can be obtained from the first line by the substitutions  $q^+ \rightarrow p_1^+ + p_2^+ - q^+$  and  $I^2, p_1^+ \leftrightarrow I^3, p_2^+$ . The complete vertex should be proportional to  $K^\wedge$ , a property not shared by the contribution of the second term. But we have not yet included the swordfish graphs, which we turn to in the next section.

We close this section by giving the  $\delta \rightarrow 0$  limit of the triangle graphs with the contributions of the second terms of (39) omitted.

$$\begin{aligned} \Gamma_{\Delta-}^{\wedge\wedge\vee} &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int dT_1 dT_2 dT_3 \delta(T_{13}q^+ - T_{12}p_1^+ - T_2p_2^+) \\ &\quad \exp \left\{ -\frac{T_1 T_3 (k_1 - k_0)^2 + T_1 T_2 (k_1 - k_2)^2 + T_2 T_3 (k_2 - k_0)^2}{T_1 + T_2 + T_3} \right\} \\ &\quad \left\{ \left[ \frac{K^2 T_1 T_2 T_3}{T_{13}^4} + \frac{p_3^+ (T_2 p_2^+ + T_3 p_1^+)}{T_{13} (T_{13} + \delta)^2} \right] \frac{A_3}{p_3^{+2}} + \left[ \frac{K^2 T_1 T_2 T_3}{T_{13}^4} + \frac{p_2^+ (T_2 p_3^+ + T_1 p_1^+)}{T_{13} (T_{13} + \delta)^2} \right] \frac{A_2}{p_3^{+2}} \right. \\ &\quad \left. + \left[ \frac{K^2 T_1 T_2 T_3}{T_{13}^4} + \frac{p_1^+ (T_3 p_3^+ + T_1 p_2^+)}{T_{13} (T_{13} + \delta)^2} \right] \frac{A_1}{p_3^{+2}} \right\} \end{aligned} \quad (43)$$

We simplify this expression by changing variables to  $T = T_{13}, x = T_1/T_{13}, y = T_2/T_{13}$  and evaluating the integral over  $T$ . Define

$$H(x, y) = x(1 - x - y)p_1^2 + xyp_2^2 + y(1 - x - y)p_3^2 \quad (44)$$

and we obtain

$$\begin{aligned} \Gamma_{\Delta-}^{\wedge\wedge\vee} &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)p_1^+ - yp_2^+) \\ &\quad \left\{ \left[ \frac{K^2 xy(1-x-y)}{H} - p_3^+ (yp_2^+ + (1-x-y)p_1^+) \ln(\delta H e^{\gamma+1}) \right] \frac{A_3}{p_3^{+2}} \right. \\ &\quad + \left[ \frac{K^2 xy(1-x-y)}{H} - p_2^+ (yp_3^+ + xp_1^+) \ln(\delta H e^{\gamma+1}) \right] \frac{A_2}{p_3^{+2}} \\ &\quad \left. + \left[ \frac{K^2 xy(1-x-y)}{H} - p_1^+ ((1-x-y)p_3^+ + xp_2^+) \ln(\delta H e^{\gamma+1}) \right] \frac{A_1}{p_3^{+2}} \right\} \end{aligned} \quad (45)$$

where we have used the  $\delta \rightarrow 0$  behavior of the integral

$$\int_0^\infty \frac{T dT}{(T + \delta)^2} e^{-HT} = I(H\delta) + H\delta I'(H\delta) \quad (46)$$

$$\sim -\ln(\delta H) - 1 - \gamma = -\ln(\delta H e^{\gamma+1}). \quad (47)$$

Finally, it is convenient to use an integration by parts to convert the  $\ln H$  terms to  $H^{-1}dH/dx$ , which enables an explicit isolation of the divergent parts of the integral. This is done by writing the coefficients of the  $\ln$ 's, after using the delta function constraint to eliminate  $y$  in favor of  $x$ , as derivatives with respect to  $x$ :

$$yp_2^+ p_3^+ + (1-x-y)p_1^+ p_3^+ = -\frac{d}{dx} [x(p_2^+(q^+ + p_1^+) - p_1^+(q^+ - p_1^+)) - x^2 p_1^+ p_2^+] \quad (48)$$

$$yp_2^+ p_3^+ + xp_1^+ p_2^+ = -\frac{d}{dx} [xp_2^+(q^+ - xp_1^+)] \quad (49)$$

$$(1-x-y)p_1^+ p_3^+ + xp_1^+ p_2^+ = -\frac{d}{dx} [xp_1^+(p_1^+ + p_2^+ - q^+ - xp_2^+)] \quad (50)$$

The delta function for general  $q^+ \neq p_1^+$  limits the range of  $x$  to  $0 < x < q^+/p_1^+$  when  $q^+ < p_1^+$  and to  $0 < x < (p_1^+ + p_2^+ - q^+)/p_2^+$  when  $q^+ > p_1^+$ . The surface terms from the integration by parts only contribute

at the upper limit. They are

$$\begin{aligned}
& \text{Surface Terms}^{\wedge\wedge\vee} \\
&= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \frac{1}{p_1^+ + p_2^+} \left\{ \sum_{q^+ < p_1^+} \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{q^+(p_1^+ + p_2^+ - q^+) A_3}{p_3^{+2}} + \frac{(p_1^+ + p_2^+) q^+ (p_1^+ - q^+) A_1}{p_1^+ p_3^{+2}} \right] \right. \\
&+ \left. \sum_{q^+ > p_1^+} \ln(\delta e^{\gamma+1} H_{>}) \left[ \frac{q^+(p_1^+ + p_2^+ - q^+) A_3}{p_3^{+2}} + \frac{(p_1^+ + p_2^+) (q^+ - p_1^+) (p_1^+ + p_2^+ - q^+) A_2}{p_2^+ p_3^{+2}} \right] \right\} \quad (51)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \frac{1}{p_1^+ + p_2^+} \left\{ \sum_{q^+ < p_1^+} \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{q^{+3}}{p_3^{+2} (p_1^+ + p_2^+ - q^+)} + \frac{(p_1^+ + p_2^+ - q^+)^3}{p_3^{+2} q^+} + \frac{p_1^+ (p_1^+ + p_2^+)}{(p_1^+ - q^+) q^+} \right] \right. \\
&+ \left. \sum_{q^+ > p_1^+} \ln(\delta e^{\gamma+1} H_{>}) \left[ \frac{q^{+3}}{p_3^{+2} (p_1^+ + p_2^+ - q^+)} + \frac{(p_1^+ + p_2^+ - q^+)^3}{p_3^{+2} q^+} + \frac{p_2^+ (p_1^+ + p_2^+)}{(q^+ - p_1^+) (p_1^+ + p_2^+ - q^+)} \right] \right\} \quad (52)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left\{ \sum_{q^+ < p_1^+} \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{2}{q^+} + \frac{1}{p_1^+ - q^+} \right] \right. \\
&+ \sum_{q^+ > p_1^+} \ln(\delta e^{\gamma+1} H_{>}) \left[ \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} + \frac{1}{q^+ - p_1^+} \right] \\
&- \sum_{q^+ < p_1^+} \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{2q^{+2} - 2q^+ (p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
&- \left. \sum_{q^+ > p_1^+} \ln(\delta e^{\gamma+1} H_{>}) \left[ \frac{2q^{+2} - 2q^+ (p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \right\} \quad (53)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) \right. \\
&+ \sum_{q^+ < p_1^+} \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{2}{q^+} + \frac{1}{p_1^+ - q^+} \right] + \sum_{q^+ > p_1^+} \left[ \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} + \frac{1}{q^+ - p_1^+} \right] \\
&+ \sum_{q^+ < p_1^+} \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^2 (p_1^+ + p_2^+)^2 (p_1^+ - q^+)}{(p_1 + p_2)^2 p_1^{+2} (p_1^+ + p_2^+ - q^+)} \\
&+ \sum_{q^+ > p_1^+} \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_2^2 (p_1^+ + p_2^+)^2 (q^+ - p_1^+)}{(p_1 + p_2)^2 p_2^{+2} q^+} \\
&- \sum_{q^+ < p_1^+} \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{2q^{+2} - 2q^+ (p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
&- \left. \sum_{q^+ > p_1^+} \ln(\delta e^{\gamma+1} H_{>}) \left[ \frac{2q^{+2} - 2q^+ (p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \right\} \quad (54) \\
&\sim -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) \right. \\
&+ \sum_{q^+} \left[ \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{2}{q^+} \right] + \sum_{q^+ \neq p_1^+} \frac{1}{|p_1^+ - q^+|} - \ln \frac{(p_1^+ + p_2^+)^2}{p_1^+ p_2^+}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} \ln \frac{p_1^2(p_1^+ + p_2^+)}{(p_1 + p_2)^2 p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} \ln \frac{p_2^2(p_1^+ + p_2^+)}{(p_1 + p_2)^2 p_2^+} \right] \\
& + \int_0^{p_1^+} dq^+ \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+)(p_1^+ - q^+)}{p_1^+(p_1^+ + p_2^+ - q^+)} \\
& + \int_{p_1^+}^{p_1^+ + p_2^+} dq^+ \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+)(q^+ - p_1^+)}{p_2^+ q^+} \\
& - \int_0^{p_1^+} dq^+ \ln(\delta e^{\gamma+1} H_<) \left[ \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
& - \int_{p_1^+}^{p_1^+ + p_2^+} dq^+ \ln(\delta e^{\gamma+1} H_>) \left[ \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \Big\} \quad (55)
\end{aligned}$$

where

$$H_< = H \left( \frac{q^+}{p_1^+} \right) = \frac{q^+}{p_1^+} \left( 1 - \frac{q^+}{p_1^+} \right) p_1^2, \quad H_> = H \left( \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) = \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \left( \frac{q^+ - p_1^+}{p_2^+} \right) p_2^2 \quad (56)$$

It is also useful to introduce

$$H_0 \equiv H(0) = \frac{q^+}{p_1^+ + p_2^+} \left( 1 - \frac{q^+}{p_1^+ + p_2^+} \right) p_3^2 \quad (57)$$

Then the last two lines can be written more compactly as

$$\begin{aligned}
& - \int_0^{p_1^+ + p_2^+} dq^+ \ln(\delta e^{\gamma+1} H_0) \left[ \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
& - \int_0^{p_1^+} dq^+ \ln(H_</H_0) \left[ \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
& - \int_{p_1^+}^{p_1^+ + p_2^+} dq^+ \ln(H_>/H_0) \left[ \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
& = - \int_0^1 du \ln(\delta e^{\gamma+1} u(1-u)p_3^2) [2u^2 - 2u + 4] \\
& - \int_0^{p_1^+ + p_2^+} dq^+ \ln \left( \frac{H(x_{max})}{H_0} \right) \left[ \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \quad (58)
\end{aligned}$$

The ultraviolet ( $\delta \rightarrow 0$ ) divergence of the triangle graph is completely contained in these surface terms. The coefficient of  $\ln \delta$  involves the sums

$$\begin{aligned}
& \sum_{q^+ < p_1^+} \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{2}{q^+} + \frac{1}{p_1^+ - q^+} - \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \\
& + \sum_{q^+ > p_1^+} \left[ \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} + \frac{1}{q^+ - p_1^+} - \frac{2q^{+2} - 2q^+(p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \rightarrow \\
& - \frac{11}{3} + \sum_{q^+ < p_1^+} \left[ \frac{1}{q^+} + \frac{1}{p_1^+ - q^+} \right] + \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} \right] + \sum_{q^+ > p_1^+} \left[ \frac{1}{q^+ - p_1^+} + \frac{1}{p_2^+ - (q^+ - p_1^+)} \right] \\
& \rightarrow 2 \ln[M_1 M_2 (M_1 + M_2)] + 6\gamma - \frac{11}{3} \quad (59)
\end{aligned}$$

where  $M_i = p_i^+/m$  is a large positive integer. In this evaluation we replace the sums by integrals for the non-singular terms. The sums are kept discrete for the singular terms with the interpretations

$$\sum_{q^+ < p_1^+} \frac{1}{q^+} = \sum_{q^+ < p_1^+} \frac{1}{p_1^+ - q^+} = \sum_{l=1}^{M_1-1} \frac{1}{l} = \psi(M_1) + \gamma \sim \ln M_1 + \gamma \quad (60)$$

$$\sum_{q^+ > p_1^+} \frac{1}{p_1^+ + p_2^+ - q^+} = \sum_{q^+ > p_1^+} \frac{1}{q^+ - p_1^+} = \sum_{l=1}^{M_2-1} \frac{1}{l} = \psi(M_2) + \gamma \sim \ln M_2 + \gamma \quad (61)$$

where  $\psi(z) = \Gamma'(z)/\Gamma(z)$  is the digamma function and  $\gamma$  is Euler's constant.

But notice that the three discrete sums as grouped in the third line of (59) are precisely those that occur in the coefficient of the log divergences of the self energies for the three legs coming into the vertex function. Furthermore the coefficients of these sums are half those in the self energies and the sign is opposite. Thus all these discrete divergent sums cancel up to half of the ones on external legs. It is also noteworthy that the summands actually cancel locally, i.e. independently for each  $q^+$ , as already pointed out in [10]. Since the worldsheet organizes loops according to their location  $\sigma$ , it is very satisfactory that a loop at fixed  $\sigma$  will not have spurious  $p^+$  divergences provided it is summed over all times.

After integration by parts the amplitude reads:

$$\begin{aligned} \Gamma_{\Delta-}^{\wedge\vee} &= \text{Surface Terms} - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)p_1^+ - yp_2^+) \\ &\quad \frac{1}{H} \left\{ \left[ K^2 xy(1-x-y) - [xp_2^+ q^+ + xp_1^+(p_1^+ + p_2^+ - q^+) - x^2 p_1^+ p_2^+] \frac{dH}{dx} \right] \frac{A_3}{p_3^{+2}} \right. \\ &\quad + \left[ K^2 xy(1-x-y) - xp_1^+(p_1^+ + p_2^+ - q^+ - xp_2^+) \frac{dH}{dx} \right] \frac{A_1}{p_3^{+2}} \\ &\quad \left. + \left[ K^2 xy(1-x-y) - xp_2^+(q^+ - xp_1^+) \frac{dH}{dx} \right] \frac{A_2}{p_3^{+2}} \right\} \quad (62) \end{aligned}$$

$$\begin{aligned} &= \text{Surface Terms} - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_0^{x_{max}} \frac{dx}{p_1^+ + p_2^+} \\ &\quad \frac{1}{H} \left\{ -x \left[ \frac{p_2^+(p_1^+ + p_2^+ - q^+)^2}{q^+ p_3^{+2}} + \frac{p_1^+ q^{+2}}{(p_1^+ + p_2^+ - q^+) p_3^{+2}} \right] \frac{dH}{dx} \right. \\ &\quad + \left[ K^2 xy(1-x-y) - xp_1^+(p_1^+ + p_2^+ - q^+ - xp_2^+) \frac{dH}{dx} \right] \frac{1}{q^{+2}} \left( \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{p_3^{+2}} \right) \\ &\quad \left. + \left[ K^2 xy(1-x-y) - xp_2^+(q^+ - xp_1^+) \frac{dH}{dx} \right] \frac{1}{(p_1^+ + p_2^+ - q^+)^2} \left( \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + \frac{q^{+2}}{p_3^{+2}} \right) \right\} \quad (63) \end{aligned}$$

$$\begin{aligned} &= \text{Surface Terms} - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_0^{x_{max}} \frac{dx}{p_1^+ + p_2^+} \\ &\quad \left\{ - \left[ \frac{p_2^+(p_1^+ + p_2^+ - q^+)^2}{q^+ p_3^{+2}} + \frac{p_1^+ q^{+2}}{(p_1^+ + p_2^+ - q^+) p_3^{+2}} \right] I_1 + \frac{1}{q^{+2}} \left( \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{p_3^{+2}} \right) I_3 \right. \\ &\quad \left. + \frac{1}{(p_1^+ + p_2^+ - q^+)^2} \left( \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + \frac{q^{+2}}{p_3^{+2}} \right) I_2 \right\} \quad (64) \end{aligned}$$

Here  $x_{max} = q^+/p_1^+$  ( $x_{max} = (p_1^+ + p_2^+ - q^+)/p_2^+$ ) if  $q^+ < p_1^+$  ( $q^+ > p_1^+$ ) respectively. Evidently the continuum limit of the  $q^+$  sums involves divergences due to the singularities in the  $A_i$  when  $q^+ \sim 0, p_1^+ + p_2^+, p_1^+$ . Although the divergences seem to be linear, cancellations soften those near 0,  $p_1^+ + p_2^+$ . The divergence near  $p_1^+$  however

is not softened in the triangle graph itself, but we shall see that the swordfish diagrams cancel the most divergent part leaving it logarithmic as well.

To show this we first note, using the delta function to eliminate  $y$ ,

$$H \rightarrow x \frac{p_1^+ + p_2^+ - q^+ - xp_2^+}{p_1^+ + p_2^+} p_1^2 + x \frac{q^+ - xp_1^+}{p_1^+ + p_2^+} p_2^2 + \frac{q^+ - xp_1^+}{p_1^+ + p_2^+} \frac{p_1^+ + p_2^+ - q^+ - xp_2^+}{p_1^+ + p_2^+} p_3^2 \quad (65)$$

$$\frac{dH}{dx} \rightarrow \frac{p_1^+ + p_2^+ - q^+ - 2xp_2^+}{p_1^+ + p_2^+} p_1^2 + \frac{q^+ - 2xp_1^+}{p_1^+ + p_2^+} p_2^2 - \frac{p_1^+(p_1^+ + p_2^+ - q^+) + q^+ p_2^+ - 2xp_1^+ p_2^+}{(p_1^+ + p_2^+)^2} p_3^2 \quad (66)$$

Then with a little rearrangement we find

$$\begin{aligned} I_1 &\equiv \frac{x}{H} \frac{dH}{dx} \\ &= 2 + \frac{x}{H} \left[ \frac{p_1^+(p_1^+ + p_2^+ - q^+)}{p_1^+ + p_2^+} \left( \frac{p_3^2}{p_1^+ + p_2^+} - \frac{p_1^2}{p_1^+} \right) + \frac{p_2^+ q^+}{p_1^+ + p_2^+} \left( \frac{p_3^2}{p_1^+ + p_2^+} - \frac{p_2^2}{p_2^+} \right) \right] \\ &\quad - \frac{2}{H} \frac{q^+(p_1^+ + p_2^+ - q^+)}{(p_1^+ + p_2^+)^2} p_3^2 \end{aligned} \quad (67)$$

$$\begin{aligned} I_2 &\equiv \frac{1}{H} \left\{ K^2 xy(1 - x - y) - [xp_2^+(q^+ - xp_1^+)] \frac{dH}{dx} \right\} \\ &= xp_1^+ p_2^+ + x \frac{q^+ - p_1^+}{H} \left[ xp_2^+ p_1^2 + (xp_1^+ - q^+) p_2^2 - \frac{p_2^+(xp_1^+ - q^+)}{p_1^+ + p_2^+} p_3^2 \right] \\ &= xp_1^+ p_2^+ - (q^+ - p_1^+)(p_1^+ + p_2^+) + \frac{(q^+ - p_1^+)(p_1^+ + p_2^+ - q^+)}{H} \left[ xp_1^2 + \frac{q^+ - xp_1^+}{p_1^+ + p_2^+} p_3^2 \right] \end{aligned} \quad (68)$$

$$\equiv xp_1^+ p_2^+ + \hat{I}_2 \quad (69)$$

$$\begin{aligned} I_3 &\equiv \frac{1}{H} \left\{ K^2 xy(1 - x - y) - [xp_1^+(p_1^+ + p_2^+ - q^+ - xp_2^+)] \frac{dH}{dx} \right\} \\ &= xp_1^+ p_2^+ + x \frac{p_1^+ - q^+}{H} \left[ xp_1^+ p_2^2 + (xp_2^+ - p_1^+ - p_2^+ + q^+) p_1^2 - \frac{p_1^+(xp_2^+ - p_1^+ - p_2^+ + q^+)}{p_1^+ + p_2^+} p_3^2 \right] \\ &= xp_1^+ p_2^+ + (q^+ - p_1^+)(p_1^+ + p_2^+) + \frac{q^+(p_1^+ - q^+)}{H} \left[ xp_2^2 + \frac{p_1^+ + p_2^+ - xp_2^+ - q^+}{p_1^+ + p_2^+} p_3^2 \right] \end{aligned} \quad (70)$$

$$\equiv xp_1^+ p_2^+ + \hat{I}_3 \quad (71)$$

We see that the second and third terms in the final expressions for  $I_2, I_3$  supply a zero at  $q^+ = p_1^+$ , softening the divergence near  $q^+ \sim p_1^+$  to a logarithmic one. The linear divergence comes entirely from the first terms.

The integrals of  $I_1, I_2, I_3$  over  $x$  are elementary, and the ones involving  $H^{-1}$  are conveniently written in terms of the roots  $r_+, r_-$  of the quadratic polynomial  $H(x) = A(x - r_+)(x - r_-)$ :

$$r_{\pm} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (72)$$

$$A = \frac{p_1^+ p_2^+}{p_1^+ + p_2^+} \left[ \frac{p_3^2}{p_1^+ + p_2^+} - \frac{p_1^2}{p_1^+} - \frac{p_2^2}{p_2^+} \right] \quad (73)$$

$$B = \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} p_1^2 + \frac{q^+}{p_1^+ + p_2^+} p_2^2 - \frac{q^+ p_2^+ + (p_1^+ + p_2^+ - q^+) p_1^+}{(p_1^+ + p_2^+)^2} p_3^2 \quad (74)$$

$$C = \frac{q^+(p_1^+ + p_2^+ - q^+)}{(p_1^+ + p_2^+)^2} p_3^2 \quad (75)$$

$$\int_0^{x_{max}} \frac{dx}{H} = \frac{1}{A(r_+ - r_-)} \left[ \ln \frac{r_+ - x_{max}}{r_+} - \ln \frac{r_- - x_{max}}{r_-} \right] \quad (76)$$

$$\int_0^{x_{max}} \frac{xdx}{H} = \frac{r_+}{A(r_+ - r_-)} \ln \frac{r_+ - x_{max}}{r_+} - \frac{r_-}{A(r_+ - r_-)} \ln \frac{r_- - x_{max}}{r_-} \quad (77)$$

We now consider in turn the behavior of these expressions near the singular values of  $q^+ = 0, p_1^+, p_1^+ + p_2^+$ . First consider  $q^+ \sim 0$ . Then  $x_{max} = q^+/p_1^+$  and we find

$$\int_0^{x_{max}} \frac{dx}{H} \sim \frac{p_1^+ + p_2^+}{(p_1^+ + p_2^+)p_1^2 - p_1^+ p_3^2} \ln \frac{(p_1^+ + p_2^+)p_1^2}{p_1^+ p_3^2} \quad \text{for } q^+ \sim 0 \quad (78)$$

$$\int_0^{x_{max}} \frac{xdx}{H} \sim \frac{q^+(p_1^+ + p_2^+)/p_1^+}{(p_1^+ + p_2^+)p_1^2 - p_1^+ p_3^2} \left[ \frac{p_1^+ p_3^2}{p_1^+ p_3^2 - (p_1^+ + p_2^+)p_1^2} \ln \frac{(p_1^+ + p_2^+)p_1^2}{p_1^+ p_3^2} + 1 \right] \quad \text{for } q^+ \sim 0 \quad (79)$$

Next we consider  $q^+ \sim p_1^+ + p_2^+$ , for which  $x_{max} = (p_1^+ + p_2^+ - q^+)/p_2^+$ . Then

$$\int_0^{x_{max}} \frac{dx}{H} \sim \frac{p_1^+ + p_2^+}{(p_1^+ + p_2^+)p_2^2 - p_2^+ p_3^2} \ln \frac{(p_1^+ + p_2^+)p_2^2}{p_2^+ p_3^2} \quad \text{for } q^+ \sim p_1^+ + p_2^+ \quad (80)$$

$$\int_0^{x_{max}} \frac{xdx}{H} \sim \frac{(p_1^+ + p_2^+ - q^+)(p_1^+ + p_2^+)}{p_2^+ [(p_1^+ + p_2^+)p_2^2 - p_2^+ p_3^2]} \left[ \frac{p_2^+ p_3^2}{p_2^+ p_3^2 - (p_1^+ + p_2^+)p_2^2} \ln \frac{(p_1^+ + p_2^+)p_2^2}{p_2^+ p_3^2} + 1 \right] \quad \text{for } q^+ \sim p_1^+ + p_2^+ \quad (81)$$

Finally, for  $q^+ \sim p_1^+$  we have to treat separately the cases  $q^+ < p_1^+$  (when  $x_{max} = q^+/p_1^+$ ) and  $q^+ > p_1^+$  (when  $x_{max} = (p_1^+ + p_2^+ - q^+)/p_2^+$ ). Actually, the singular factor multiplying these integrals is just  $(q^+ - p_1^+)^{-1}$ . Since the  $q^+$  sum is symmetric about  $q^+ = p_1^+$ , a divergence will occur only because of a discontinuity in the summand due to the different behavior of  $x_{max}$  on either side:

$$x_{max} = \begin{cases} q^+/p_1^+ \sim 1 + \delta q^+/p_1^+ & \text{for } q^+ < p_1^+ \\ (p_1^+ + p_2^+ - q^+)/p_2^+ \sim 1 - \delta q^+/p_2^+ & \text{for } q^+ > p_1^+ \end{cases} \quad (82)$$

where  $\delta q^+ = q^+ - p_1^+ \ll p_1^+$ . Then we find

$$\int_0^{x_{max}} \frac{dx}{H} \sim -\frac{1}{D} \left[ \ln \frac{|\delta q^+|}{D} - \ln \frac{(p_1^+ + p_2^+)^2 D}{p_1^+ p_2^+ p_3^2} \right] - \frac{1}{D} \begin{cases} \ln \frac{p_1^2}{p_1^+} & \text{for } q^+ < p_1^+ \\ \ln \frac{p_2^2}{p_2^+} & \text{for } q^+ > p_1^+ \end{cases} \quad \text{for } q^+ \sim p_1^+ \quad (83)$$

$$\int_0^{x_{max}} \frac{xdx}{H} \sim -\frac{1}{D} \left[ \ln \frac{|\delta q^+|}{D} - \frac{A+D}{A} \ln \frac{(p_1^+ + p_2^+)^2 D}{p_1^+ p_2^+ p_3^2} \right] - \frac{1}{D} \begin{cases} \ln \frac{p_1^2}{p_1^+} & \text{for } q^+ < p_1^+ \\ \ln \frac{p_2^2}{p_2^+} & \text{for } q^+ > p_1^+ \end{cases} \quad \text{for } q^+ \sim p_1^+ \quad (84)$$

$$D \equiv \frac{p_1^+ p_2^+}{p_1^+ + p_2^+} \left[ \frac{p_1^2}{p_1^+} + \frac{p_2^2}{p_2^+} \right] \quad (85)$$

We see that the discontinuity (value for  $q^+ < p_1^+$  - value for  $q^+ > p_1^+$ ) of either of the right sides about the point  $q^+ = p_1^+$  is simply

$$-\frac{1}{D} \ln \frac{p_1^2 p_2^+}{p_2^2 p_1^+} = -\frac{p_1^+ + p_2^+}{p_1^+ p_2^2 + p_2^+ p_1^2} \ln \frac{p_1^2 p_2^+}{p_2^2 p_1^+} \quad (86)$$

Now we combine these results to extract the remaining divergences in  $\Gamma_{\Delta-}^{\wedge\wedge\vee}$ . First of all we find that

$$\int dx I_1 \sim \begin{cases} \frac{q^+}{p_1^+} \left[ 1 + \frac{p_1^+ p_3^2}{p_1^+ p_3^2 - (p_1^+ + p_2^+)p_1^2} \ln \frac{(p_1^+ + p_2^+)p_1^2}{p_1^+ p_3^2} \right] & \text{for } q^+ \sim 0 \\ \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \left[ 1 + \frac{p_2^+ p_3^2}{p_2^+ p_3^2 - (p_1^+ + p_2^+)p_2^2} \ln \frac{(p_1^+ + p_2^+)p_2^2}{p_2^+ p_3^2} \right] & \text{for } q^+ \sim p_1^+ + p_2^+ \end{cases} \quad (87)$$

The singular factor multiplying this integral has only simple poles at  $q^+ = 0$  and  $q^+ = p_1^+ + p_2^+$  which are killed by the above behavior, so the  $I_1$  term in  $\Gamma$  is finite in the continuum limit.

The  $I_3$  term and the  $I_2$  term involve singular factors  $q^{+-2}$ ,  $(q^+ - p_1^+)^{-2}$  and  $(p_1^+ + p_2^+ - q^+)^{-2}$ ,  $(q^+ - p_1^+)^{-2}$  respectively. We first examine the singular behavior for  $q^+ \sim 0$  which is found only in the  $I_3$  term. We find

$$\int dx I_3 \sim -q^+(p_1^+ + p_2^+) + q^+ p_1^+ \frac{p_3^2(p_1^+ + p_2^+)}{(p_1^+ + p_2^+)p_1^2 - p_1^+ p_3^2} \ln \frac{(p_1^+ + p_2^+)p_1^2}{p_1^+ p_3^2} \quad \text{for } q^+ \sim 0 \quad (88)$$

So the singular contribution of the triangle graph at  $q^+ = 0$  reads

$$-\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \frac{2}{q^+} \left[ -1 + \frac{p_1^+ p_3^2}{(p_1^+ + p_2^+)p_1^2 - p_1^+ p_3^2} \ln \frac{(p_1^+ + p_2^+)p_1^2}{p_1^+ p_3^2} \right] \quad (89)$$

Very similarly, the singular behavior of the triangle graph near  $q^+ = p_1^+ + p_2^+$ , which comes from the  $I_2$  term, reads

$$-\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \frac{2}{p_1^+ + p_2^+ - q^+} \left[ -1 + \frac{p_2^+ p_3^2}{(p_1^+ + p_2^+)p_2^2 - p_2^+ p_3^2} \ln \frac{(p_1^+ + p_2^+)p_2^2}{p_2^+ p_3^2} \right] \quad (90)$$

Finally we separate the singular contributions near  $q^+ = p_1^+$ . These are found in both the  $I_2$  and  $I_3$  terms. Unlike the previous contributions, there are both linear and logarithmic divergence near  $q^+ = p_1^+$  in the triangle graph. Fortunately, the linear divergence comes only from the first terms of  $I_2$  and  $I_3$ , and those terms did not contribute to the divergences near  $q^+ = 0, p_1^+ + p_2^+$ , so it is convenient to evaluate their contribution completely (i.e. not just the singular parts).

$$\begin{aligned} \text{First Term} &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ + p_2^+} K^\wedge \sum_{q^+} \int_0^{x_{max}} dx x \frac{A_1 + A_2}{p_3^{+2}} \\ &= \frac{g^3}{8\pi^2} K^\wedge \sum_{q^+} x_{max}^2 \left[ \frac{1}{q^{+2}} \left( \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{p_3^{+2}} \right) + \frac{1}{(p_1^+ + p_2^+ - q^+)^2} \left( \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + \frac{q^{+2}}{p_3^{+2}} \right) \right] \\ &= \frac{g^3}{8\pi^2} K^\wedge \sum_{q^+ < p_1^+} \frac{1}{p_1^{+2}} \left[ \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{p_3^{+2}} + \frac{q^{+2}}{(p_1^+ + p_2^+ - q^+)^2} \left( \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + \frac{q^{+2}}{p_3^{+2}} \right) \right] \\ &\quad + \frac{g^3}{8\pi^2} K^\wedge \sum_{q^+ > p_1^+} \frac{1}{p_2^{+2}} \left[ \frac{(p_1^+ + p_2^+ - q^+)^2}{q^{+2}} \left( \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + \frac{(p_1^+ + p_2^+ - q^+)^2}{p_3^{+2}} \right) + \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + \frac{q^{+2}}{p_3^{+2}} \right] \\ &= \frac{g^3}{8\pi^2} K^\wedge \left[ \sum_{q^+ \neq p_1^+} \frac{2}{(q^+ - p_1^+)^2} + (p_1^+ + p_2^+) \sum_{q^+ < p_1^+} \frac{q^+ p_2^+ + p_1^+ (p_1^+ + p_2^+ - q^+)}{(q^+ - p_1^+) p_1^{+2} (p_1^+ + p_2^+ - q^+)^2} \right. \\ &\quad + \sum_{q^+ < p_1^+} \frac{(p_1^+ + p_2^+ - q^+)^4 + q^{+4}}{p_1^{+2} p_3^{+2} (p_1^+ + p_2^+ - q^+)^2} + (p_1^+ + p_2^+) \sum_{q^+ > p_1^+} \frac{q^+ p_2^+ + p_1^+ (p_1^+ + p_2^+ - q^+)}{(p_1^+ - q^+) q^{+2} p_2^{+2}} \\ &\quad \left. + \sum_{q^+ > p_1^+} \frac{(p_1^+ + p_2^+ - q^+)^4 + q^{+4}}{p_2^{+2} p_3^{+2} q^{+2}} \right] \\ &= \frac{g^3}{8\pi^2} K^\wedge \left[ \sum_{q^+ \neq p_1^+} \left\{ \frac{2}{(q^+ - p_1^+)^2} - \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} \frac{2}{|q^+ - p_1^+|} \right\} \right. \\ &\quad \left. + \sum_{q^+ < p_1^+} \left\{ \frac{2(p_1^+ + p_2^+)}{p_1^+ p_2^+ (p_1^+ + p_2^+ - q^+)} + \frac{(p_1^+ + p_2^+ - q^+)^4 + q^{+4} + p_3^{+4}}{p_1^{+2} p_3^{+2} (p_1^+ + p_2^+ - q^+)^2} \right\} \right] \end{aligned}$$



$$+ \sum_{q^+ > p_1^+} \left\{ \frac{2(p_1^+ + p_2^+)}{p_1^+ p_2^+ q^+} + \frac{(p_1^+ + p_2^+ - q^+)^4 + q^{+4} + p_3^{+4}}{p_2^{+2} p_3^{+2} q^{+2}} \right\} \Big] \quad (91)$$

$$\rightarrow \frac{g^3}{8\pi^2} K^\wedge \left[ \sum_{q^+ \neq p_1^+} \left\{ \frac{2}{(q^+ - p_1^+)^2} - \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} \frac{2}{|q^+ - p_1^+|} \right\} + 2 \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} \left\{ \ln \frac{(p_1^+ + p_2^+)^2}{p_1^+ p_2^+} + 4 \right\} \right. \\ \left. + \frac{2}{3} \frac{1}{p_1^+ + p_2^+} - 4 \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} \left( \frac{p_1^+}{p_2^+} \ln \frac{p_1^+ + p_2^+}{p_2^+} + \frac{p_2^+}{p_1^+} \ln \frac{p_1^+ + p_2^+}{p_1^+} \right) \right] \quad (92)$$

Where we have evaluated the continuum limit of the convergent terms. The first term in square brackets is canceled by a corresponding term from the swordfish diagrams (see Eq. (112,113,114) in the following section). Borrowing from (112) we find

$$\text{First Term} + \Gamma_{\text{SF}^+}^{\wedge\wedge\vee} \rightarrow \frac{g^3}{8\pi^2} K^\wedge \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} \left[ - \sum_{q^+ \neq p_1^+} \frac{2}{|q^+ - p_1^+|} + \frac{50}{3} - 4 \left( \frac{p_2^+}{p_1^+} + \frac{p_1^+}{p_2^+} - \frac{1}{2} \right) \ln \frac{(p_1^+ + p_2^+)^2}{p_1^+ p_2^+} \right] \\ + \frac{g^3}{12\pi^2} \frac{K^\wedge}{p_1^+ + p_2^+} - \frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge \quad (93)$$

The rest of  $I_2$  and  $I_3$  contribute a logarithmic divergence near  $q^+ = p_1^+$ . As already mentioned, since the  $q^+$  sum is symmetric about  $q^+ = p_1^+$ , the singular behavior depends on the discontinuity:

$$\frac{1}{2} \text{Disc} \int dx \hat{I}_2 \sim -\frac{1}{2} (q^+ - p_1^+) \frac{p_2^+ (p_1^+ + p_2^+) p_1^2}{p_1^+ p_2^+ + p_2^+ p_1^2} \ln \frac{p_2^+ p_1^2}{p_1^+ p_2^2} \quad (94)$$

$$\frac{1}{2} \text{Disc} \int dx \hat{I}_3 \sim +\frac{1}{2} (q^+ - p_1^+) \frac{p_1^+ (p_1^+ + p_2^+) p_2^2}{p_1^+ p_2^+ + p_2^+ p_1^2} \ln \frac{p_2^+ p_1^2}{p_1^+ p_2^2} \quad (95)$$

These two contributions to the divergence of the vertex function combine to

$$-\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_0^{x_{\max}} \frac{dx}{p_1^+ + p_2^+} \frac{\hat{I}_2 + \hat{I}_3}{(q^+ - p_1^+)^2} \sim -\frac{g^3}{8\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ \neq p_1^+} \frac{1}{|q^+ - p_1^+|} \frac{p_2^+ p_1^2 - p_1^+ p_2^2}{p_1^+ p_2^+ + p_2^+ p_1^2} \ln \frac{p_2^+ p_1^2}{p_1^+ p_2^2} \quad (96)$$

Let us now combine (89), (90), (93), and (96), with the surface terms (55):

$$-\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) \right. \\ \left. + \sum_{q^+} \left\{ \frac{1}{q^+} \left[ \frac{(p_1^+ + p_2^+) p_1^2 + p_1^+ p_3^2}{(p_1^+ + p_2^+) p_1^2 - p_1^+ p_3^2} \ln \frac{(p_1^+ + p_2^+) p_1^2}{p_1^+ p_3^2} \right] + \frac{1}{p_1^+ + p_2^+ - q^+} \left[ \frac{(p_1^+ + p_2^+) p_2^2 + p_2^+ p_3^2}{(p_1^+ + p_2^+) p_2^2 - p_2^+ p_3^2} \ln \frac{(p_1^+ + p_2^+) p_2^2}{p_2^+ p_3^2} \right] \right\} \right. \\ \left. + \frac{1}{2} \sum_{q^+ \neq p_1^+} \frac{1}{|q^+ - p_1^+|} \frac{p_2^+ p_1^2 - p_1^+ p_2^2}{p_1^+ p_2^+ + p_2^+ p_1^2} \ln \frac{p_2^+ p_1^2}{p_1^+ p_2^2} + \frac{25}{3} - 2 \left( \frac{p_2^+}{p_1^+} + \frac{p_1^+}{p_2^+} \right) \ln \frac{(p_1^+ + p_2^+)^2}{p_1^+ p_2^+} \right. \\ \left. + \int_0^{p_1^+} dq^+ \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+) (p_1^+ - q^+)}{p_1^+ (p_1^+ + p_2^+ - q^+)} \right. \\ \left. + \int_{p_1^+}^{p_1^+ + p_2^+} dq^+ \left[ \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+) (q^+ - p_1^+)}{p_2^+ q^+} \right. \\ \left. - \int_0^{p_1^+} dq^+ \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{2q^{+2} - 2q^+ (p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \right. \\ \left. - \int_{p_1^+}^{p_1^+ + p_2^+} dq^+ \ln(\delta e^{\gamma+1} H_{>}) \left[ \frac{2q^{+2} - 2q^+ (p_1^+ + p_2^+) + 4(p_1^+ + p_2^+)^2}{(p_1^+ + p_2^+)^3} \right] \right\}$$

$$+ \frac{g^3}{12\pi^2} \frac{K^\wedge}{p_1^+ + p_2^+} - \frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge \quad (97)$$

## 5.2 Swordfish Graphs

We shall see that the swordfish graphs are nominally  $O(\delta)$  and the integrals that define them give  $O(1)$  only in the region where all  $T_i = O(\delta)$ . Consequently, they are linear polynomials in the transverse momenta. Specifically, there are three distinct graphs, labeled according to the external leg attached to the cubic vertex:

$$\begin{aligned} \Gamma_{SF3}^{\wedge\wedge\vee} &= -\frac{g^3}{8\pi^2} \sum_{q^+} \frac{p_1^+ + p_2^+}{q^+(p_1^+ + p_2^+ - q^+)} \left[ 1 - \frac{(p_1^+ + q^+)(q^+ - p_1^+ - 2p_2^+)}{(p_1^+ - q^+)^2} \right] (x_3 k_2 + (1 - x_3) k_0)^\wedge \\ &\quad \int dT \frac{\delta}{(T + \delta)^2} \exp \left\{ -x_3(1 - x_3)T(k_2 - k_0)^2 - \frac{\delta T}{T + \delta} (x_3 \mathbf{k}_2 + (1 - x_3) \mathbf{k}_0)^2 \right\} \end{aligned} \quad (98)$$

$$\begin{aligned} &\rightarrow -\frac{g^3}{8\pi^2} \sum_{q^+} \frac{p_1^+ + p_2^+}{q^+(p_1^+ + p_2^+ - q^+)} \left[ 1 - \frac{(p_1^+ + q^+)(q^+ - p_1^+ - 2p_2^+)}{(p_1^+ - q^+)^2} \right] (x_3 k_2 + (1 - x_3) k_0)^\wedge \\ &= -\frac{g^3}{8\pi^2} \frac{1}{p_1^+ + p_2^+} \sum_{q^+} \frac{1}{x_3(1 - x_3)} \left[ 1 - \frac{(x_3 + \eta)(x_3 + \eta - 2)}{(\eta - x_3)^2} \right] (x_3 k_2 + (1 - x_3) k_0)^\wedge \end{aligned} \quad (99)$$

Here  $x_3 = q^+/(p_1^+ + p_2^+)$  and  $\eta = p_1^+/(p_1^+ + p_2^+)$ .

$$\begin{aligned} \Gamma_{SF1}^{\wedge\wedge\vee} &= -\frac{g^3}{8\pi^2} \sum_{q^+ < p_1^+} \left\{ \frac{q^+}{p_1^+(p_1^+ - q^+)} \left[ -2 - \frac{(p_1^+ + p_2^+ + q^+)(q^+ - p_1^+ + p_2^+)}{(p_1^+ + p_2^+ - q^+)^2} - \frac{(2q^+ - p_1^+)(p_1^+ + 2p_2^+)}{p_1^{+2}} \right] \right. \\ &\quad \left. + \frac{p_1^+ - q^+}{p_1^+ q^+} \left[ 1 - \frac{(2q^+ - p_1^+)(p_1^+ + 2p_2^+)}{p_1^{+2}} \right] \right\} (x_1 k_1 + (1 - x_1) k_0)^\wedge \\ &\quad \int dT \frac{\delta}{(T + \delta)^2} \exp \left\{ -x_1(1 - x_1)T(k_1 - k_0)^2 - \frac{\delta T}{T + \delta} (x_1 \mathbf{k}_1 + (1 - x_1) \mathbf{k}_0)^2 \right\} \\ &\rightarrow -\frac{g^3}{8\pi^2} \sum_{q^+ < p_1^+} \left\{ \frac{q^+}{p_1^+(p_1^+ - q^+)} \left[ -3 - \frac{(p_1^+ + p_2^+ + q^+)(q^+ - p_1^+ + p_2^+)}{(p_1^+ + p_2^+ - q^+)^2} \right] \right. \\ &\quad \left. + \left[ \frac{q^+}{p_1^+(p_1^+ - q^+)} + \frac{p_1^+ - q^+}{p_1^+ q^+} \right] \left[ 1 - \frac{(2q^+ - p_1^+)(p_1^+ + 2p_2^+)}{p_1^{+2}} \right] \right\} (x_1 k_1 + (1 - x_1) k_0)^\wedge \\ &= -\frac{g^3}{8\pi^2} \frac{1}{p_1^+} \sum_{q^+ < p_1^+} \left\{ \frac{x_1}{1 - x_1} \left[ -3 - \frac{(x_1 + \eta^{-1})(x_1 + \eta^{-1} - 2)}{(\eta^{-1} - x_1)^2} \right] \right. \\ &\quad \left. + \left[ \frac{x_1}{1 - x_1} + \frac{1 - x_1}{x_1} \right] [1 - (2x_1 - 1)(2\eta^{-1} - 1)] \right\} (x_1 k_1 + (1 - x_1) k_0)^\wedge \end{aligned} \quad (100)$$

This time  $x_1 = q^+/p_1^+$ .

$$\begin{aligned} \Gamma_{SF2}^{\wedge\wedge\vee} &= -\frac{g^3}{8\pi^2} \sum_{q^+ > p_1^+} \left\{ \frac{p_1^+ + p_2^+ - q^+}{p_2^+(q^+ - p_1^+)} \left[ -2 - \frac{(q^+ - 2p_1^+ - 2p_2^+)(q^+ - 2p_1^+)}{q^{+2}} \right. \right. \\ &\quad \left. \left. - \frac{(2p_1^+ + p_2^+ - 2q^+)(2p_1^+ + p_2^+)}{p_2^{+2}} \right] \right. \\ &\quad \left. + \frac{q^+ - p_1^+}{p_2^+(p_1^+ + p_2^+ - q^+)} \left[ 1 - \frac{(2p_1^+ + p_2^+ - 2q^+)(2p_1^+ + p_2^+)}{p_2^{+2}} \right] \right\} (x_2 k_1 + (1 - x_2) k_2)^\wedge \\ &\quad \int dT \frac{\delta}{(T + \delta)^2} \exp \left\{ -x_2(1 - x_2)T(k_1 - k_2)^2 - \frac{\delta T}{T + \delta} (x_2 \mathbf{k}_1 + (1 - x_2) \mathbf{k}_2)^2 \right\} \end{aligned} \quad (102)$$

$$\begin{aligned}
& \rightarrow -\frac{g^3}{8\pi^2} \sum_{q^+ > p_1^+} \left\{ \frac{p_1^+ + p_2^+ - q^+}{p_2^+(q^+ - p_1^+)} \left[ -3 - \frac{(q^+ - 2p_1^+ - 2p_2^+)(q^+ - 2p_1^+)}{q^{+2}} \right] \right. \\
& \quad \left. + \left[ \frac{p_1^+ + p_2^+ - q^+}{p_2^+(q^+ - p_1^+)} + \frac{q^+ - p_1^+}{p_2^+(p_1^+ + p_2^+ - q^+)} \right] \left[ 1 - \frac{(2p_1^+ + p_2^+ - 2q^+)(2p_1^+ + p_2^+)}{p_2^{+2}} \right] \right\} (x_2 k_1 + (1 - x_2) k_2)^\wedge \\
& = -\frac{g^3}{8\pi^2} \frac{1}{p_2^+} \sum_{q^+ > p_1^+} \left\{ \frac{x_2}{1 - x_2} \left[ -3 - \frac{(x_2 + (1 - \eta)^{-1})(x_2 + (1 - \eta)^{-1} - 2)}{(x_2 - (1 - \eta)^{-1})^2} \right] \right. \\
& \quad \left. + \left[ \frac{x_2}{1 - x_2} + \frac{1 - x_2}{x_2} \right] [1 - (2x_2 - 1)(2(1 - \eta)^{-1} - 1)] \right\} (x_2 k_1 + (1 - x_2) k_2)^\wedge \quad (103)
\end{aligned}$$

Finally  $x_2 = (p_1^+ + p_2^+ - q^+)/p_2^+$ . The arrows indicate the  $\delta \rightarrow 0$  limit of each result, which as promised is seen to be linear in the transverse momenta, though not simply proportional to  $K^\wedge$ . The dependence on longitudinal momenta is far from simple. However these complicated expressions combine nicely with the contributions of the the second terms of (39) discussed at the end of the previous subsection. Indeed collecting together all of the singular terms from those and the swordfish diagrams shows that they are all proportional to  $K^\wedge$ :

$$\begin{aligned}
\text{Singular}(\wedge \wedge \vee) &= \frac{g^3}{8\pi^2} K^\wedge \left\{ \sum_{q^+ < p_1^+} \left[ \frac{2(p_1^+ + p_2^+)^2}{p_1^{+2}(p_1^+ + p_2^+ - q^+)^2} - \frac{4(p_1^+ + p_2^+)}{p_1^{+2}(p_1^+ + p_2^+ - q^+)} - \frac{2}{(q^+ - p_1^+)^2} \right] \right. \\
& \quad \left. + \sum_{q^+ > p_1^+} \left[ \frac{2(p_1^+ + p_2^+)^2}{p_2^{+2}q^{+2}} - \frac{4(p_1^+ + p_2^+)}{p_2^{+2}q^+} - \frac{2}{(q^+ - p_1^+)^2} \right] \right\} \quad (104)
\end{aligned}$$

The corresponding singular contributions for the other spin configurations are

$$\begin{aligned}
\text{Singular}(\wedge \vee \wedge) &= \frac{g^3}{8\pi^2} K^\wedge \left\{ \sum_{q^+ < p_1^+} \left[ \frac{2p_2^{+2}}{p_1^{+2}(p_1^+ + p_2^+ - q^+)^2} - \frac{4p_2^{+2}}{p_1^{+2}(p_1^+ + p_2^+ - q^+)} - \frac{2p_2^{+2}}{p_3^{+2}(q^+ - p_1^+)^2} \right] \right. \\
& \quad \left. - \sum_{q^+ \neq p_1^+} \frac{4p_2^{+2}}{p_3^{+2}(p_1^+ - q^+)} + \sum_{q^+ > p_1^+} \left[ \frac{2}{q^{+2}} - \frac{2p_2^{+2}}{p_3^{+2}((q^+ - p_1^+)^2)} \right] \right\} \quad (105)
\end{aligned}$$

$$\begin{aligned}
\text{Singular}(\vee \wedge \wedge) &= \frac{g^3}{8\pi^2} K^\wedge \left\{ \sum_{q^+ < p_1^+} \left[ \frac{2}{(p_1^+ + p_2^+ - q^+)^2} - \frac{2p_1^{+2}}{p_3^{+2}(q^+ - p_1^+)^2} \right] \right. \\
& \quad \left. - \sum_{q^+ \neq p_1^+} \frac{4p_1^{+2}}{p_3^{+2}(q^+ - p_1^+)} + \sum_{q^+ > p_1^+} \left[ \frac{2p_1^{+2}}{p_2^{+2}q^{+2}} - \frac{4p_1^{+2}}{p_2^{+2}q^+} - \frac{2p_1^{+2}}{p_3^{+2}(q^+ - p_1^+)^2} \right] \right\} \quad (106)
\end{aligned}$$

In addition to these there are polynomials in  $q^+$  which depend separately on the  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_0$ :

$$\begin{aligned}
& \frac{g^3}{8\pi^2} \sum_{q^+ < p_1^+} \left[ k'^\wedge \left( \frac{2q^{+2}}{p_1^+(p_1^+ + p_2^+)^2} + \frac{4}{p_1^+} \right) + k^\wedge \left( \frac{q^{+2}}{p_1^{+2}} \left( \frac{2}{p_1^+ + p_2^+} - \frac{4(p_1^+ + 2p_2^+)}{p_1^{+2}} \right) - \frac{4(p_1^+ + p_2^+)}{p_1^{+2}} + \frac{4q^+ p_2^+}{p_1^{+3}} \right) \right. \\
& \quad \left. + k''^\wedge \left( \frac{q^{+2}}{p_1^{+2}} \left( \frac{4(p_1^+ + 2p_2^+)}{p_1^{+2}} - \frac{2(2p_1^+ + p_2^+)}{(p_1^+ + p_2^+)^2} \right) + \frac{4q^+}{p_1^+} \left( \frac{1}{p_1^+ + p_2^+} - \frac{(p_1^+ + 3p_2^+)}{p_1^{+2}} \right) + \frac{8p_2^+}{p_1^{+2}} \right) \right] \quad (107)
\end{aligned}$$

plus a sum over  $q^+ > p_1^+$  whose summand is obtained from the above by the substitutions  $q^+ \rightarrow p_1^+ + p_2^+ - q^+$ ,  $\mathbf{k}_2, p_1^+ \leftrightarrow \mathbf{k}_0, p_2^+$ . Since these summands are nonsingular it is safe to replace the sums by integrals and perform them, after which spectacular simplification takes place:

$$\text{Polynomial Contribution}(\wedge \wedge \vee) \rightarrow \frac{g^3}{8\pi^2} \left[ \frac{14}{3} \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} K^\wedge - \frac{2}{3} (k_1 + k_2 + k_0)^\wedge \right] \quad (108)$$

The coefficients break down as  $14/3 = 26/3 - 4$ ,  $-2/3 = -4/3 + 2/3$ , with the first terms coming from the swordfish graphs while the second ones come from the triangle graphs. The only thing that changes in the analogous contribution for the other spin configurations is the coefficient of  $K^\wedge$ , which just matches the tree coefficient:

$$\text{Polynomial Contribution}(\wedge \vee \wedge) \rightarrow \frac{g^3}{8\pi^2} \left[ \frac{14}{3} \frac{p_2^+}{p_1^+(p_1^+ + p_2^+)} K^\wedge - \frac{2}{3} (k_1 + k_2 + k_0)^\wedge \right] \quad (109)$$

$$\text{Polynomial Contribution}(\vee \wedge \wedge) \rightarrow \frac{g^3}{8\pi^2} \left[ \frac{14}{3} \frac{p_1^+}{p_2^+(p_1^+ + p_2^+)} K^\wedge - \frac{2}{3} (k_1 + k_2 + k_0)^\wedge \right] \quad (110)$$

The second term in the square brackets of each of these contributions is the only regularization artifact that will require a new counter-term, beyond the usual coupling, self-energy, and wave function renormalization. It is spin independent and can be given a local worldsheet interpretation if we simply rewrite it in the form

$$-\frac{g^3}{12\pi^2} [(k_2^\wedge - k_1^\wedge) + (k_0^\wedge - k_1^\wedge) + 3k_1^\wedge] \quad (111)$$

The first two terms can be produced by appropriate insertions of  $\partial q^\wedge / \partial \sigma$  near the interaction point on the worldsheet, and the last term is already local since  $k$  is the value of  $q^\wedge$  at the interaction point.

In summary the contribution of the swordfish diagrams combined with the delta terms from the triangle diagrams is given by

$$\begin{aligned} \Gamma_{\text{SF}+}^{\wedge\wedge\vee} \rightarrow & \frac{g^3}{8\pi^2} K^\wedge \frac{p_1^+ + p_2^+}{p_1^+ p_2^+} \left[ \frac{26}{3} - \frac{4p_2^+}{p_1^+} \ln \frac{p_1^+ + p_2^+}{p_2^+} - \frac{4p_1^+}{p_2^+} \ln \frac{p_1^+ + p_2^+}{p_1^+} \right] \\ & - \frac{g^3}{8\pi^2} K^\wedge \sum_{q^+ \neq p_1^+} \frac{2}{(q^+ - p_1^+)^2} - \frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge \end{aligned} \quad (112)$$

$$\begin{aligned} \Gamma_{\text{SF}+}^{\wedge\vee\wedge} \rightarrow & \frac{g^3}{8\pi^2} K^\wedge \frac{p_2^+}{p_1^+(p_1^+ + p_2^+)} \left[ \frac{26}{3} - \frac{4(p_1^+ + p_2^+)}{p_1^+} \ln \frac{p_1^+ + p_2^+}{p_2^+} - \frac{4p_1^+}{p_1^+ + p_2^+} \ln \frac{p_1^+}{p_2^+} \right] \\ & - \frac{g^3}{8\pi^2} K^\wedge \sum_{q^+ \neq p_1^+} \frac{2p_2^{+2}}{p_3^{+2}(q^+ - p_1^+)^2} - \frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge \end{aligned} \quad (113)$$

$$\begin{aligned} \Gamma_{\text{SF}+}^{\vee\wedge\wedge} \rightarrow & \frac{g^3}{8\pi^2} K^\wedge \frac{p_1^+}{p_2^+(p_1^+ + p_2^+)} \left[ \frac{26}{3} - \frac{4p_2^+}{p_1^+ + p_2^+} \ln \frac{p_2^+}{p_1^+} - \frac{4(p_1^+ + p_2^+)}{p_2^+} \ln \frac{p_1^+ + p_2^+}{p_1^+} \right] \\ & - \frac{g^3}{8\pi^2} K^\wedge \sum_{q^+ \neq p_1^+} \frac{2p_1^{+2}}{p_3^{+2}(q^+ - p_1^+)^2} - \frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge \end{aligned} \quad (114)$$

The arrows signify that the sums over discretized  $q^+$  have been replaced by integrals and performed wherever possible. The only term where this is not possible is shown as a discretized sum. As mentioned in the previous section, this term cancels a corresponding term in the triangle diagram calculation.

### 5.3 Renormalization at One Loop

When we studied wave function renormalization, we found that the log divergence had a divergent  $p^+$  dependent coefficient. But then we found that this  $p^+$  dependence was exactly canceled by corresponding contributions from the triangle vertex corrections. Thus we can drop the  $p^+$  dependence and use the wave function renormalization constant (18):

$$Z_3 = 1 - \frac{g^2 N_c}{4\pi^2} \left( \frac{11}{6} \right) \ln(\mu^2 \delta). \quad (115)$$

We can similarly drop the  $p^+$  dependent part of the log divergence in the vertex renormalization and use the vertex renormalization constant

$$\frac{1}{Z_1} = 1 + \frac{g^2 N_c}{8\pi^2} \left( \frac{11}{3} \right) \ln(\mu^2 \delta) \quad (116)$$

We can now write the relation between renormalized and bare coupling

$$\begin{aligned} g_R &= g \frac{Z^{3/2}}{Z_1} \\ &= g \left( 1 + \frac{g^2 N_c}{8\pi^2} \left( \frac{11}{3} - \frac{3}{2} \frac{11}{3} \right) \ln(\mu^2 \delta) \right) \\ &= g \left( 1 - \frac{11}{3} \frac{g^2 N_c}{16\pi^2} \ln(\mu^2 \delta) \right) \end{aligned} \quad (117)$$

and for the Callan-Symanzik beta function

$$\beta(g) \equiv \mu \frac{dg_R}{d\mu} = -\frac{11g^3 N_c}{24\pi^2} + O(g^5). \quad (118)$$

This is the known result for the beta function for *planar* Yang-Mills field theory.

## 5.4 Three gluon vertex contribution to scattering of Glue by Glue

The one-loop three gluon vertex contribution to the four gluon scattering amplitude requires putting two of the three gluons on shell, and there are three distinct cases. First we put  $p_1^2 = p_3^2 = 0$ , so we have

$$\begin{aligned} H &\rightarrow xyp_2^2, \quad K^2 \rightarrow p_1^+(p_1^+ + p_2^+)p_2^2 \\ \Gamma_{\Delta-}^{\wedge\wedge\vee} &\rightarrow -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)p_1^+ - yp_2^+) \left\{ \right. \\ &\quad p_1^+(p_1^+ + p_2^+)(1-x-y) \frac{A_1 + A_2 + A_3}{p_3^{+2}} \\ &\quad - p_3^+(yp_2^+ + (1-x-y)p_1^+) \ln(\delta xyp_2^2 e^{\gamma+1}) \frac{A_3}{p_3^{+2}} - p_2^+(yp_3^+ + xp_1^+) \ln(\delta xyp_2^2 e^{\gamma+1}) \frac{A_2}{p_3^{+2}} \\ &\quad \left. - p_1^+((1-x-y)p_3^+ + xp_2^+) \ln(\delta xyp_2^2 e^{\gamma+1}) \frac{A_1}{p_3^{+2}} \right\} \\ &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_0^{x_m} \frac{dx}{p_1^+ + p_2^+} \left\{ p_1^+(p_1^+ + p_2^+ - q^+ - xp_2^+) \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right. \\ &\quad \left. \left[ p_2^+ q^+ \frac{A_2 + A_3}{p_3^{+2}} + p_1^+(p_1^+ + p_2^+ - q^+) \frac{A_1 + A_3}{p_3^{+2}} - 2xp_1^+ p_2^+ \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right] \ln(\delta xyp_2^2 e^{\gamma+1}) \right\} \end{aligned} \quad (120)$$

Formulas that enable the explicit evaluation of the  $x, y$  integrals are listed in an appendix. Then

$$\begin{aligned} \Gamma_{\Delta-}^{\wedge\wedge\vee} &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ < p_1^+} \left\{ B_0 + B_1 \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right\} \\ &\quad - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ > p_1^+} \left\{ B'_0 + B'_1 \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) + B_2 \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \\ &\quad - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ \neq p_1^+} \left\{ B_1 \ln \frac{q^+}{p_1^+ + p_2^+} \right\}, \quad \text{for } p_1^2 = p_3^2 = 0 \end{aligned} \quad (121)$$

Where the  $B_i$  are given by

$$\begin{aligned}
B_0 &= \frac{q^+}{p_1^+ + p_2^+} \left( p_1^+ + p_2^+ - q^+ - \frac{q^+}{2p_1^+} p_2^+ \right) \frac{A_2}{p_3^{+2}} - \frac{q^{+2} p_2^+}{2p_1^+ (p_1^+ + p_2^+)} \frac{A_3 - A_1}{p_3^{+2}} \\
B_1 &= \frac{q^+ (p_1^+ + p_2^+ - q^+)}{p_1^+ + p_2^+} \frac{A_3}{p_3^{+2}} + \frac{q^+ (p_1^+ - q^+)}{p_1^+} \frac{A_1}{p_3^{+2}} \\
B'_0 &= \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \left( q^+ - \frac{p_1^+ + p_2^+ - q^+}{2p_2^+} p_1^+ \right) \frac{A_1}{p_3^{+2}} - \frac{(p_1^+ + p_2^+ - q^+)^2 p_1^+}{2p_2^+ (p_1^+ + p_2^+)} \frac{A_3 - A_2}{p_3^{+2}} \\
B'_1 &= \frac{q^+ (p_1^+ + p_2^+ - q^+)}{p_1^+ + p_2^+} \frac{A_3}{p_3^{+2}} + \frac{(p_1^+ + p_2^+ - q^+) (q^+ - p_1^+)}{p_2^+} \frac{A_2}{p_3^{+2}} \\
B_2 &= \frac{q^+ (q^+ - p_1^+)}{p_1^+} \frac{A_1}{p_3^{+2}} + \frac{(p_1^+ + p_2^+ - q^+) (q^+ - p_1^+)}{p_2^+} \frac{A_2}{p_3^{+2}}
\end{aligned} \tag{122}$$

The various spin configurations are obtained by substituting the appropriate expressions for  $A_1, A_2, A_3$ :

$$\begin{aligned}
B_0^{\wedge\wedge\vee} &= \frac{p_1^+ p_2^+}{p_1^+ + p_2^+} \frac{1}{(q^+ - p_1^+)^2} - \frac{p_2^+}{2p_1^+ (p_1^+ + p_2^+)^3} \frac{q^{+4} + (p_1^+ + p_2^+)^4 + (p_1^+ + p_2^+ - q^+)^4}{(p_1^+ + p_2^+ - q^+)^2} \\
B_0'^{\wedge\wedge\vee} &= \frac{p_1^+ p_2^+}{p_1^+ + p_2^+} \frac{1}{(q^+ - p_1^+)^2} - \frac{p_1^+}{2p_2^+ (p_1^+ + p_2^+)^3} \frac{q^{+4} + (p_1^+ + p_2^+)^4 + (p_1^+ + p_2^+ - q^+)^4}{q^{+2}} \\
B_0^{\wedge\vee\wedge} &= \frac{p_2^{+2}}{(p_1^+ + p_2^+)^2} \left[ \frac{p_1^+ [p_2^+ + 2(p_1^+ - q^+)]}{(p_1^+ + p_2^+) (q^+ - p_1^+)^2} + p_1^+ \frac{(q^+ - p_1^+)^2 + 2p_2^+ (p_1^+ - q^+) + 3p_2^{+2}}{p_2^{+3} (p_1^+ + p_2^+)} \right. \\
&\quad \left. - \frac{(p_1^+ + p_2^+) [(p_1^+ - q^+)^4 + p_2^{+4} + (p_1^+ + p_2^+ - q^+)^4]}{2p_1^+ p_2^{+3} (p_1^+ + p_2^+ - q^+)^2} \right] \\
B_0'^{\wedge\vee\wedge} &= \frac{p_2^{+2}}{(p_1^+ + p_2^+)^2} \left[ \frac{p_1^+ [p_2^+ + 2(p_1^+ - q^+)]}{(p_1^+ + p_2^+) (q^+ - p_1^+)^2} + p_1^+ \frac{(q^+ - p_1^+)^2 + 2p_2^+ (p_1^+ - q^+) + 3p_2^{+2}}{p_2^{+3} (p_1^+ + p_2^+)} \right. \\
&\quad \left. - \frac{(p_1^+ + p_2^+) p_1^+}{p_2^{+2} q^{+2}} \right] \\
B_0^{\vee\wedge\wedge} &= \frac{p_1^{+2}}{(p_1^+ + p_2^+)^2} \left[ \frac{p_2^+ [p_1^+ + 2(q^+ - p_1^+)]}{(p_1^+ + p_2^+) (q^+ - p_1^+)^2} + p_2^+ \frac{(q^+ - p_1^+)^2 + 2p_1^+ (q^+ - p_1^+) + 3p_1^{+2}}{p_1^{+3} (p_1^+ + p_2^+)} \right. \\
&\quad \left. - \frac{(p_1^+ + p_2^+) p_2^+}{p_1^+ (p_1^+ + p_2^+ - q^+)^2} \right] \\
B_0'^{\vee\wedge\wedge} &= \frac{p_1^{+2}}{(p_1^+ + p_2^+)^2} \left[ \frac{p_2^+ [p_1^+ + 2(q^+ - p_1^+)]}{(p_1^+ + p_2^+) (q^+ - p_1^+)^2} + p_2^+ \frac{(q^+ - p_1^+)^2 + 2p_1^+ (q^+ - p_1^+) + 3p_1^{+2}}{p_1^{+3} (p_1^+ + p_2^+)} \right. \\
&\quad \left. - \frac{(p_1^+ + p_2^+) [(p_1^+ - q^+)^4 + p_1^{+4} + q^{+4}]}{2p_2^+ p_1^{+3} q^{+2}} \right]
\end{aligned} \tag{123}$$

It is worthwhile to immediately integrate these rather unwieldy expressions for  $B_0, B'_0$  and combine them with the corresponding swordfish amplitude:

$$\begin{aligned}
\Gamma_{\text{S.F.}}^{\wedge\wedge\vee} - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( \sum_{q^+ < p_1^+} B_0 + \sum_{q^+ > p_1^+} B'_0 \right) &= \frac{g^3}{12\pi^2} \left[ -\frac{p_3^+}{p_1^+ p_2^+} K^\wedge + \frac{K^\wedge}{p_3^+} - (k_1 + k_2 + k_0)^\wedge \right] \\
\Gamma_{\text{S.F.}}^{\wedge\vee\wedge} - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( \sum_{q^+ < p_1^+} B_0 + \sum_{q^+ > p_1^+} B'_0 \right) &= \frac{g^3}{12\pi^2} \left[ -\frac{p_2^+}{p_1^+ p_3^+} K^\wedge + \frac{K^\wedge}{p_2^+} - (k_1 + k_2 + k_0)^\wedge \right]
\end{aligned}$$

$$\Gamma_{\text{S.F.}}^{\vee\wedge\wedge} - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( \sum_{q^+ < p_1^+} B_0 + \sum_{q^+ > p_1^+} B'_0 \right) = \frac{g^3}{12\pi^2} \left[ -\frac{p_1^+}{p_3^+ p_2^+} K^\wedge + \frac{K^\wedge}{p_1^+} - (k_1 + k_2 + k_0)^\wedge \right] \quad (124)$$

Note here that one spin configuration can be obtained from another by suitably cycling the indices. As noted later  $B_0, B'_0$  enter the triangle amplitude in the same way for all choices of pairs of on-shell external lines. This will not be the case for the other  $B$ 's.

$$\begin{aligned} B_1^{\wedge\wedge\vee} &= \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} + \frac{2q^+(p_1^+ + p_2^+ - q^+)}{(p_1^+ + p_2^+)^3} - \frac{4}{p_1^+ + p_2^+} \\ B_1^{\wedge\vee\wedge} &= \frac{p_2^{+2}}{p_3^{+2}} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \\ B_1^{\vee\wedge\wedge} &= \frac{p_1^{+2}}{p_3^{+2}} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} + \frac{2q^+(p_1^+ - q^+)}{p_1^{+3}} - \frac{4}{p_1^+} \right] \\ B_1^{\prime\wedge\wedge\vee} &= \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} + \frac{2q^+(p_1^+ + p_2^+ - q^+)}{(p_1^+ + p_2^+)^3} - \frac{4}{p_1^+ + p_2^+} \\ B_1^{\prime\wedge\vee\wedge} &= \frac{p_2^{+2}}{p_3^{+2}} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} + \frac{2(q^+ - p_1^+)(p_1^+ + p_2^+ - q^+)}{p_2^{+3}} - \frac{4}{p_2^+} \right] \\ B_1^{\vee\vee\wedge} &= \frac{p_1^{+2}}{p_3^{+2}} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \\ B_2^{\wedge\wedge\vee} &= \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \\ B_2^{\wedge\vee\wedge} &= \frac{p_2^{+2}}{p_3^{+2}} \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} + \frac{2(q^+ - p_1^+)(p_1^+ + p_2^+ - q^+)}{p_2^{+3}} - \frac{4}{p_2^+} \right] \\ B_2^{\vee\wedge\wedge} &= \frac{p_1^{+2}}{p_3^{+2}} \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} + \frac{2(q^+ - p_1^+)q^+}{p_1^{+3}} + \frac{4}{p_1^+} \right] \end{aligned} \quad (125)$$

$$\begin{aligned} \Gamma_{\Delta}^{\wedge\wedge\vee} &= \Gamma_{\Delta-}^{\wedge\wedge\vee} + \Gamma_{\text{S.F.},+}^{\wedge\wedge\vee} \\ &= \frac{g^3}{12\pi^2} \left( -\frac{p_3^+}{p_1^+ p_2^+} K^\wedge - \frac{K^\wedge}{p_1^+ + p_2^+} - (k_1 + k_2 + k_0)^\wedge \right) \\ &\quad - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( \sum_{q^+ \neq p_1^+} B_1 \ln \frac{q^+}{p_1^+ + p_2^+} + \sum_{q^+ < p_1^+} B_1 \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right. \\ &\quad \left. + \sum_{q^+ > p_1^+} \left\{ B'_1 \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) + B_2 \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \right), \quad \text{for } p_1^2 = p_3^2 = 0 \end{aligned} \quad (126)$$

The final simplification step is to explicitly evaluate the  $q^+$  summation for the polynomial parts of the  $B$ 's:

$$\begin{aligned} \Gamma_{\Delta}^{\wedge\wedge\vee} &= \frac{g^3}{12\pi^2} \left[ -\frac{70}{3} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge - (k_1 + k_2 + k_0)^\wedge \right] - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( -\frac{11}{3} \ln(\delta p_2^2 e^\gamma) \right. \\ &\quad \left. + \sum_{q^+ \neq p_1^+} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \right. \\ &\quad \left. + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right\} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
& \quad \left. + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\}, \quad \text{for } p_1^2 = p_3^2 = 0 \quad (127)
\end{aligned}$$

For the other two spin configurations we merely quote the final answers:

$$\begin{aligned}
\Gamma_{\Delta}^{\wedge \vee \wedge} &= \frac{g^3}{12\pi^2} \left[ -\frac{70}{3} \frac{p_2^+}{p_1^+ p_3^+} K^\wedge + \frac{K^\wedge}{p_2^+} - (k_1 + k_2 + k_0)^\wedge \right] - \frac{g^3}{4\pi^2} \frac{p_2^+}{p_1^+ p_3^+} K^\wedge \left( -\frac{11}{3} \ln(\delta p_2^2 e^\gamma) \right. \\
&+ \sum_{q^+ \neq p_1^+} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
&+ \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right\} \\
&+ \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
&\quad \left. + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \Bigg) \\
\Gamma_{\Delta}^{\vee \wedge \wedge} &= \frac{g^3}{12\pi^2} \left[ -\frac{70}{3} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge - (k_1 + k_2 + k_0)^\wedge \right] - \frac{g^3}{4\pi^2} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge \left( -\frac{11}{3} \ln(\delta p_2^2 e^\gamma) \right. \\
&+ \sum_{q^+ \neq p_1^+} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
&+ \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right\} \\
&+ \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
&\quad \left. + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \Bigg), \quad \text{for } p_1^2 = p_3^2 = 0 \quad (128)
\end{aligned}$$

We note that apart from suitable relabeling of indices in passing from one spin configuration to another, there is a breaking of the cyclic symmetry through putting legs 1, 3 on shell. The term  $K/p_i^+$  in the square brackets on first line of each case is uncanceled when the on-shell lines have like helicity and canceled otherwise.

Next we choose the on-shell pair  $p_1^2 = p_2^2 = 0$  and obtain

$$\begin{aligned}
H &\rightarrow y(1-x-y)p_3^2, \quad K^2 \rightarrow -p_1^+ p_2^+ p_3^2 \\
\Gamma_{\Delta-}^{\wedge \wedge \vee} &\rightarrow -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)p_1^+ - yp_2^+) \left\{ -p_1^+ p_2^+ x \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right. \\
&\quad - p_3^+ (yp_2^+ + (1-x-y)p_1^+) \ln(\delta y(1-x-y)p_3^2 e^{\gamma+1}) \frac{A_3}{p_3^{+2}} \\
&\quad - p_2^+ (yp_3^+ + xp_1^+) \ln(\delta y(1-x-y)p_3^2 e^{\gamma+1}) \frac{A_2}{p_3^{+2}} \\
&\quad \left. - p_1^+ ((1-x-y)p_3^+ + xp_2^+) \ln(\delta y(1-x-y)p_3^2 e^{\gamma+1}) \frac{A_1}{p_3^{+2}} \right\} \quad (129)
\end{aligned}$$



$$\begin{aligned}
&= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_0^{x_m} \frac{dx}{p_1^+ + p_2^+} \left\{ -p_1^+ p_2^+ x \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right. \\
&+ \left. \left[ p_2^+ q^+ \frac{A_2 + A_3}{p_3^{+2}} + p_1^+ (p_1^+ + p_2^+ - q^+) \frac{A_1 + A_3}{p_3^{+2}} - 2x p_1^+ p_2^+ \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right] \ln(\delta y (1-x-y) p_3^2 e^{\gamma+1}) \right\} \quad (130)
\end{aligned}$$

Integrating over  $x, y$  then yields

$$\begin{aligned}
\Gamma_{\Delta-}^{\wedge\wedge\vee} &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ < p_1^+} \left\{ B_0 + B_1 \ln(\delta p_3^2 e^\gamma) - B_2 \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
&- \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ > p_1^+} \left\{ B'_0 + B'_1 \ln(\delta p_3^2 e^\gamma) + B_2 \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \\
&- \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ \neq p_1^+} \left\{ B'_1 \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} + B_1 \ln \frac{q^+}{p_1^+ + p_2^+} \right\}, \quad \text{for } p_1^2 = p_2^2 = 0 \quad (131)
\end{aligned}$$

Adding in the swordfish diagram and simplifying

$$\begin{aligned}
\Gamma_{\Delta-}^{\wedge\wedge\vee} &= \Gamma_{\Delta-}^{\wedge\wedge\vee} + \Gamma_{\text{S.F.}+}^{\wedge\wedge\vee} \\
&= -\frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( \frac{70}{9} - \frac{p_1^+ p_2^+}{3p_3^{+2}} - \frac{11}{3} \ln(\delta p_3^2 e^\gamma) \right. \\
&+ \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{q^+}{p_1^+ + p_2^+} \right) \right. \\
&+ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \\
&+ \left. \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
&+ \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right) \right. \\
&+ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
&+ \left. \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \Bigg), \quad \text{for } p_1^2 = p_2^2 = 0 \quad (132)
\end{aligned}$$

The other spin configurations are

$$\begin{aligned}
\Gamma_{\Delta-}^{\wedge\vee\wedge} &= -\frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_2^+}{p_1^+ p_3^+} K^\wedge \left( \frac{70}{9} - \frac{11}{3} \ln(\delta p_3^2 e^\gamma) \right. \\
&+ \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{q^+}{p_1^+ + p_2^+} \right) \right. \\
&+ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \\
&+ \left. \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
&+ \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
& + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \Bigg\} \Bigg), \quad \text{for } p_1^2 = p_2^2 = 0 \quad (133) \\
\Gamma_{\Delta}^{\vee\wedge\wedge} &= -\frac{g^3}{12\pi^2} (k_1 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge \left( \frac{70}{9} - \frac{11}{3} \ln(\delta p_3^2 e^\gamma) \right. \\
& + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{q^+}{p_1^+ + p_2^+} \right) \right. \\
& + \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \\
& + \left. \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
& + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right) \right. \\
& + \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
& + \left. \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\} \Bigg), \quad \text{for } p_1^2 = p_2^2 = 0 \quad (134)
\end{aligned}$$

Finally, we choose  $p_2^2 = p_3^2 = 0$ ,

$$\begin{aligned}
H &\rightarrow x(1-x-y)p_1^2, \quad K^2 \rightarrow p_2^+(p_1^+ + p_2^+)p_2^2 \quad (135) \\
\Gamma_{\Delta-}^{\wedge\wedge\vee} &\rightarrow -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)p_1^+ - yp_2^+) \left\{ p_2^+(p_1^+ + p_2^+)y \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right. \\
& - p_3^+(yp_2^+ + (1-x-y)p_1^+) \ln(\delta x(1-x-y)p_1^2 e^{\gamma+1}) \frac{A_3}{p_3^{+2}} \\
& - p_2^+(yp_3^+ + xp_1^+) \ln(\delta x(1-x-y)p_1^2 e^{\gamma+1}) \frac{A_2}{p_3^{+2}} \\
& \left. - p_1^+((1-x-y)p_3^+ + xp_2^+) \ln(\delta x(1-x-y)p_1^2 e^{\gamma+1}) \frac{A_1}{p_3^{+2}} \right\} \\
& = -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+} \int_0^{x_m} \frac{dx}{p_1^+ + p_2^+} \left\{ p_2^+(q^+ - xp_1^+) \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right. \\
& + \left[ p_2^+ q^+ \frac{A_2 + A_3}{p_3^{+2}} + p_1^+(p_1^+ + p_2^+ - q^+) \frac{A_1 + A_3}{p_3^{+2}} - 2xp_1^+ p_2^+ \frac{A_1 + A_2 + A_3}{p_3^{+2}} \right] \ln(\delta x(1-x-y)p_1^2 e^{\gamma+1}) \Bigg\} \quad (136)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\Delta-}^{\wedge\wedge\vee} &= -\frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ < p_1^+} \left\{ B_0 + B_1 \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) - B_2 \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
& - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ > p_1^+} \left\{ B'_0 + B'_1 \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right\} \\
& - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \sum_{q^+ \neq p_1^+} \left\{ B'_1 \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right\}, \quad \text{for } p_2^2 = p_3^2 = 0 \quad (137)
\end{aligned}$$

Adding in the swordfish diagram and simplifying gives

$$\begin{aligned}
\Gamma_{\Delta}^{\wedge\wedge\vee} &= \Gamma_{\Delta-}^{\wedge\wedge\vee} + \Gamma_{\text{S.F.}+}^{\wedge\wedge\vee} \\
&= -\frac{g^3}{12\pi^2}(k_1 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_3^+}{p_1^+ p_2^+} K^\wedge \left( \frac{70}{9} - \frac{11}{3} \ln(\delta p_1^2 e^\gamma) \right. \\
&\quad + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right. \\
&\quad + \left. \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
&\quad + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
&\quad + \left. \left. \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right\} \right), \quad \text{for } p_2^2 = p_3^2 = 0 \quad (138)
\end{aligned}$$

The other two spin configurations are then:

$$\begin{aligned}
\Gamma_{\Delta}^{\wedge\vee\wedge} &= -\frac{g^3}{12\pi^2}(k_1 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_2^+}{p_1^+ p_3^+} K^\wedge \left( \frac{70}{9} - \frac{11}{3} \ln(\delta p_1^2 e^\gamma) \right. \\
&\quad + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right. \\
&\quad + \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \Big\} \\
&\quad + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
&\quad + \left. \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right\} \Bigg), \quad \text{for } p_2^2 = p_3^2 = 0 \quad (139)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{\Delta}^{\vee\wedge\wedge} &= -\frac{g^3}{12\pi^2}(k_1 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge \left( \frac{70}{9} - \frac{p_2^+ p_3^+}{3p_1^{+2}} - \frac{11}{3} \ln(\delta p_1^2 e^\gamma) \right. \\
&\quad + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right. \\
&\quad + \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \Big\} \\
&\quad + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
&\quad + \left. \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right\} \Bigg), \quad \text{for } p_2^2 = p_3^2 = 0 \quad (140)
\end{aligned}$$

All of the triangle amplitudes listed in this subsection are appropriate to two incoming and one outgoing particle,  $p_1^+, p_2^+ > 0$ . We get the case of two outgoing particles by applying the dictionary (31). In particular when we assemble the triangle contributions to the scattering of glue by glue, there are four contributing diagrams in which, respectively, the gluon pairs (1, 2), (2, 3), (3, 4), (4, 1) hook onto the triangle sub-diagram.

If  $p_1^+ + p_4^+ > 0$  the first two have two incoming gluons and the last two have two outgoing gluons. When  $p_1^+ + p_4^+ < 0$  it is the first and last which have two incoming gluons.

As an example take the triangle diagram attached to gluons (34), with  $p_3^+, p_4^+ < 0$ . Then, for example,

$$\begin{aligned}
\Gamma^{\vee\wedge\wedge}(p_{12}, p_3, p_4; k_0, k_2, k_3) &= \Gamma^{\wedge\wedge\vee}(-p_4, -p_3, -p_{12}; k_0, k_3, k_2) \\
&= -\frac{g^3}{12\pi^2}(k_3 + k_2 + k_0)^\wedge - \frac{g^3}{4\pi^2} \frac{p_{12}^+}{p_3^+ p_4^+} K_{34}^\wedge \left( \frac{70}{9} - \frac{p_3^+ p_4^+}{3p_{12}^{+2}} - \frac{11}{3} \ln(\delta p_{12}^2 e^\gamma) \right. \\
&\quad + \sum_{q^+ < |p_4^+|} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{|p_4^+| - q^+} \right] \left( \ln(\delta p_{12}^2 e^\gamma) + \ln \frac{q^+}{p_1^+ + p_2^+} \right) \right. \\
&\quad + \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - |p_4^+|} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \\
&\quad + \left. \left[ \frac{2}{|p_4^+| - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{|p_4^+| - q^+}{|p_4^+|} \right\} \\
&\quad + \sum_{q^+ > |p_4^+|} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - |p_4^+|} \right] \left( \ln(\delta p_{12}^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right) \right. \\
&\quad + \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{|p_4^+| - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
&\quad + \left. \left[ \frac{2}{q^+ - |p_4^+|} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - |p_4^+|}{|p_3^+|} \right\} \Bigg), \quad \text{for } p_3^2 = p_4^2 = 0
\end{aligned} \tag{141}$$

In summary, all of the 2 like-helicity one loop cubic vertices can be put in the form

$$\Gamma_{1 \text{ loop}} = -\frac{g^3}{12\pi^2} \sum_i k_i - \frac{g^2}{8\pi^2} \Gamma_{\text{tree}} \left( \frac{70}{9} - \frac{11}{3} \ln(\delta p_o^2 e^\gamma) + S \right) + \alpha \frac{g^3}{12\pi^2} \frac{K}{p_o^+} \tag{142}$$

where the vectors  $k_i, K$  carry the polarization of the two like-helicity gluons,  $p_o$  is the four-momentum of the off-shell gluon, and  $\alpha = 1$  when the on-shell gluons have like-helicity, and  $\alpha = 0$  otherwise. Finally  $S$  is an infrared sensitive term that depends on the location of the off-shell gluon, but not on any of the gluon helicities. In the case  $p_1^+, p_2^+ > 0$ :

$$\begin{aligned}
S_1 &= \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right. \\
&\quad + \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \Bigg\} \\
&\quad + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_1^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right. \\
&\quad + \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \Bigg\} \\
S_2 &= \sum_{q^+ \neq p_1^+} \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
&\quad + \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{q^+}{p_1^+} \right) \right\} \\
&\quad + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_2^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_2^+} \right) \right\}
\end{aligned} \tag{143}$$

$$\begin{aligned}
& + \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \Big\} \\
S_3 = & \sum_{q^+ < p_1^+} \left\{ \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{q^+}{p_1^+ + p_2^+} \right) \right. \\
& + \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \\
& + \left. \left[ \frac{2}{p_1^+ - q^+} - \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} \right\} \\
& + \sum_{q^+ > p_1^+} \left\{ \left[ \frac{1}{q^+} + \frac{2}{p_1^+ + p_2^+ - q^+} + \frac{1}{q^+ - p_1^+} \right] \left( \ln(\delta p_3^2 e^\gamma) + \ln \frac{p_1^+ + p_2^+ - q^+}{p_1^+ + p_2^+} \right) \right. \\
& + \left[ \frac{2}{q^+} + \frac{1}{p_1^+ + p_2^+ - q^+} + \frac{1}{p_1^+ - q^+} \right] \ln \frac{q^+}{p_1^+ + p_2^+} \\
& + \left. \left[ \frac{2}{q^+ - p_1^+} + \frac{1}{p_1^+ + p_2^+ - q^+} - \frac{1}{q^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} \right\}
\end{aligned} \tag{145}$$

## 6 Four Point Vertex Function

### 6.1 Box Diagrams

The simplest spin configuration is all like helicity, for definiteness take  $\wedge \wedge \wedge \wedge$ . The box is the only one-loop 1PIR diagram contributing to this process. Fig. 1 shows one of the two diagrams for this process and the assignment of dual momentum variables to it. Because there are no divergences for this process, we can

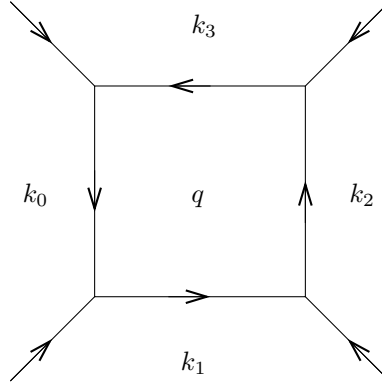


Figure 1: A typical box diagram showing the assignment of dual momenta.

immediately write down the Schwinger representation for it:

$$\begin{aligned}
\Gamma^{\wedge \wedge \wedge \wedge} = & 16g^4 \sum_{q^+} \int dT_1 dT_2 dT_3 dT_4 \delta(T_{14}q^+ - T_{24}p_1^+ - T_{34}p_2^+ - T_{44}p_3^+) \int \frac{d^2 q}{(2\pi)^3} \\
& \exp \left\{ - \frac{\delta(\mathbf{k}_0 T_1 + \mathbf{k}_1 T_2 + \mathbf{k}_2 T_3 + \mathbf{k}_3 T_4)^2}{T_{14}(T_{14} + \delta)} - (T_{14} + \delta) \mathbf{q}^2 \right\} \\
& \exp \left\{ - \frac{T_1 T_2 p_1^2 + T_1 T_3 (p_1 + p_2)^2 + T_1 T_4 p_4^2 + T_2 T_3 p_2^2 + T_2 T_4 (p_2 + p_3)^2 + T_3 T_4 p_3^2}{T_{14}} \right\}
\end{aligned}$$

$$(q + K_1)^\wedge (q + K_2)^\wedge (q + K_3)^\wedge (q + K_4)^\wedge \quad (146)$$

$$\sim \frac{2g^4}{\pi^2} \int \frac{dT_1 dT_2 dT_3 dT_4}{T_{14}^6 p_1^+ p_2^+ p_3^+ p_4^+} (T_3 K_{12}^\wedge + T_4 K_{41}^\wedge) (T_4 K_{23}^\wedge + T_1 K_{12}^\wedge) (T_2 K_{23}^\wedge + T_1 K_{34}^\wedge) (T_3 K_{34}^\wedge + T_2 K_{41}^\wedge) \\ \exp \left\{ -\frac{T_1 T_2 p_1^2 + T_1 T_3 (p_1 + p_2)^2 + T_1 T_4 p_4^2 + T_2 T_3 p_2^2 + T_2 T_4 (p_2 + p_3)^2 + T_3 T_4 p_3^2}{T_{14}} \right\} \quad (147)$$

Where we have taken  $\delta \rightarrow 0$  and  $q^+$  continuous in the last line. Actually, this box diagram is even finite on shell so one can safely set  $p_i^2 = 0$  from the beginning as well. For completeness we quote the  $K_i$  with  $\delta > 0$  even though for this box it is safe to put  $\delta = 0$  from the beginning:

$$K_1 = \frac{T_3 K_{12} + T_4 K_{41}}{T_{14} p_1^+} - \frac{\delta}{T_{14}(T_{14} + \delta)} (T_1 k_0 + T_2 k_1 + T_3 k_2 + T_4 k_3) \quad (148)$$

$$K_2 = \frac{T_4 K_{23} + T_1 K_{12}}{T_{14} p_2^+} - \frac{\delta}{T_{14}(T_{14} + \delta)} (T_1 k_0 + T_2 k_1 + T_3 k_2 + T_4 k_3) \quad (149)$$

$$K_3 = \frac{T_2 K_{23} + T_1 K_{34}}{T_{14} p_3^+} - \frac{\delta}{T_{14}(T_{14} + \delta)} (T_1 k_0 + T_2 k_1 + T_3 k_2 + T_4 k_3) \quad (150)$$

$$K_4 = \frac{T_3 K_{34} + T_2 K_{41}}{T_{14} p_4^+} - \frac{\delta}{T_{14}(T_{14} + \delta)} (T_1 k_0 + T_2 k_1 + T_3 k_2 + T_4 k_3) \quad (151)$$

The next simplest spin configuration is three like and one unlike helicity, e.g.  $\wedge \wedge \wedge \vee$ . There are also other 1PIR diagrams involving no more than one quartic vertex contributing to this process. As in the case of all like helicity this process is free of both IR and UV divergences. However, this will require cancellations against the reducible diagrams involving triangle sub-graphs. The box diagrams combine to:

$$\Gamma^{\wedge \wedge \wedge \vee} = 8g^4 \sum_{q^+} \int dT_1 dT_2 dT_3 dT_4 \delta(T_{14} q^+ - T_{24} p_1^+ - T_{34} p_2^+ - T_4 p_3^+) \int \frac{d^2 q}{(2\pi)^3} \\ \exp \left\{ -\frac{\delta(\mathbf{k}_0 T_1 + \mathbf{k}_1 T_2 + \mathbf{k}_2 T_3 + \mathbf{k}_3 T_4)^2}{T_{14}(T_{14} + \delta)} - (T_{14} + \delta) \mathbf{q}^2 \right\} \\ \exp \left\{ -\frac{T_1 T_2 p_1^2 + T_1 T_3 (p_1 + p_2)^2 + T_1 T_4 p_4^2 + T_2 T_3 p_2^2 + T_2 T_4 (p_2 + p_3)^2 + T_3 T_4 p_3^2}{T_{14}} \right\} \\ \left[ \left( \frac{(q^+ + p_4^+)^2}{q^{+2}} + \frac{q^{+2}}{(q^+ + p_4^+)^2} \right) (q + K_1)^\wedge (q + K_2)^\wedge (q + K_3)^\wedge (q + K_4)^\vee \right. \\ + \frac{p_1^{+2} p_4^{+2}}{q^{+2} (q^+ - p_1^+)^2} (q + K_1)^\vee (q + K_2)^\wedge (q + K_3)^\wedge (q + K_4)^\wedge \\ + \frac{p_2^{+2} p_4^{+2}}{(p_1^+ + p_2^+ - q^+)^2 (q^+ - p_1^+)^2} (q + K_1)^\wedge (q + K_2)^\vee (q + K_3)^\wedge (q + K_4)^\wedge \\ + \left. \frac{p_3^{+2} p_4^{+2}}{(p_1^+ + p_2^+ - q^+)^2 (q^+ + p_4^+)^2} (q + K_1)^\wedge (q + K_2)^\wedge (q + K_3)^\vee (q + K_4)^\wedge \right] \quad (152) \\ \sim \frac{g^4}{\pi^2} \sum_{q^+} \int \frac{dT_1 dT_2 dT_3 dT_4}{T_{14}} \delta(T_{14} q^+ - T_{24} p_1^+ - T_{34} p_2^+ - T_4 p_3^+) \\ \exp \left\{ -\frac{T_1 T_2 p_1^2 + T_1 T_3 (p_1 + p_2)^2 + T_1 T_4 p_4^2 + T_2 T_3 p_2^2 + T_2 T_4 (p_2 + p_3)^2 + T_3 T_4 p_3^2}{T_{14}} \right\} \\ \left[ \left( \frac{(q^+ + p_4^+)^2}{q^{+2}} + \frac{q^{+2}}{(q^+ + p_4^+)^2} \right) \left( K_1^\wedge K_2^\wedge K_3^\wedge K_4^\vee + \frac{K_1^\wedge K_2^\wedge + K_2^\wedge K_3^\wedge + K_1^\wedge K_3^\wedge}{2T_{14}} \right) \right. \\ + \left. \frac{p_1^{+2} p_4^{+2}}{q^{+2} (q^+ - p_1^+)^2} \left( K_1^\vee K_2^\wedge K_3^\wedge K_4^\wedge + \frac{K_2^\wedge K_3^\wedge + K_3^\wedge K_4^\wedge + K_2^\wedge K_4^\wedge}{2T_{14}} \right) \right]$$

$$\begin{aligned}
& + \frac{p_2^{+2} p_4^{+2}}{(p_1^+ + p_2^+ - q^+)^2 (q^+ - p_1^+)^2} \left( K_1^\wedge K_2^\vee K_3^\wedge K_4^\wedge + \frac{K_1^\wedge K_3^\wedge + K_3^\wedge K_4^\wedge + K_1^\wedge K_4^\wedge}{2T_{14}} \right) \\
& + \frac{p_3^{+2} p_4^{+2}}{(p_1^+ + p_2^+ - q^+)^2 (q^+ + p_4^+)^2} \left( K_1^\wedge K_2^\wedge K_3^\vee K_4^\wedge + \frac{K_1^\wedge K_2^\wedge + K_2^\wedge K_4^\wedge + K_1^\wedge K_4^\wedge}{2T_{14}} \right) \Big] \quad (153)
\end{aligned}$$

Here we set  $\delta \rightarrow 0$  because of the absence of UV divergences. In particular the  $K_i$  may also be taken with  $\delta = 0$ . However we leave  $q^+$  discrete, since the  $q^+$  divergences will only cancel after including the triangle sub-diagrams. Indeed the next task is to extract these divergences and show the cancellation.

The  $q^+$  divergences occur at  $q^+ = 0, p_1^+, -p_4^+, p_1^+ + p_2^+$ . The range of is  $0 < q^+ < p_1^+ p_2^+$ , so there are two endpoint singularities and two interior ones. Although the worst divergences seem to be linear, we know that those will cancel against terms from the quartic triangle diagrams, leaving at worst log divergences. We begin by scaling the variables  $T_i = x_i T_{14}$  and integrating over  $T_{14}$ :

$$\begin{aligned}
\Gamma^{\wedge\wedge\wedge\vee} & \sim \frac{g^4}{\pi^2} \sum_{q^+} \int_{x_2+x_3+x_4 \leq 1} dx_2 dx_3 dx_4 \delta(q^+ - x_{24} p_1^+ - x_{34} p_2^+ - x_4 p_3^+) \\
& \left[ \left( \frac{(q^+ + p_4^+)^2}{q^{+2}} + \frac{q^{+2}}{(q^+ + p_4^+)^2} \right) \left( \frac{K_1^\wedge K_2^\wedge K_3^\wedge K_4^\vee}{H^2} + \frac{K_1^\wedge K_2^\wedge + K_2^\wedge K_3^\wedge + K_1^\wedge K_3^\wedge}{2H} \right) \right. \\
& + \frac{p_1^{+2} p_4^{+2}}{q^{+2} (q^+ - p_1^+)^2} \left( \frac{K_1^\vee K_2^\wedge K_3^\wedge K_4^\wedge}{H^2} + \frac{K_2^\wedge K_3^\wedge + K_3^\wedge K_4^\wedge + K_2^\wedge K_4^\wedge}{2H} \right) \\
& + \frac{p_2^{+2} p_4^{+2}}{(p_1^+ + p_2^+ - q^+)^2 (q^+ - p_1^+)^2} \left( \frac{K_1^\wedge K_2^\vee K_3^\wedge K_4^\wedge}{H^2} + \frac{K_1^\wedge K_3^\wedge + K_3^\wedge K_4^\wedge + K_1^\wedge K_4^\wedge}{2H} \right) \\
& \left. + \frac{p_3^{+2} p_4^{+2}}{(p_1^+ + p_2^+ - q^+)^2 (q^+ + p_4^+)^2} \left( \frac{K_1^\wedge K_2^\wedge K_3^\vee K_4^\wedge}{H^2} + \frac{K_1^\wedge K_2^\wedge + K_2^\wedge K_4^\wedge + K_1^\wedge K_4^\wedge}{2H} \right) \right] \quad (154)
\end{aligned}$$

$$H = (1 - x_2 - x_3 - x_4)[x_2 p_1^2 + x_3(p_1 + p_2)^2 + x_4 p_4^2] + x_2 x_3 p_2^2 + x_2 x_4 (p_2 + p_3)^2 + x_3 x_4 p_3^2 \quad (155)$$

The endpoint singularities are the easiest to analyze because the delta function drastically shrinks the range of the  $x_i$ . For  $q^+ \rightarrow 0$ ,  $x_2, x_3, x_4 = O(q^+)$ , and, holding the  $p_i^2 \neq 0$ , we have

$$H \sim x_2 p_1^2 + x_3 (p_1 + p_2)^2 + x_4 p_4^2 = O(q^+) \quad (156)$$

$$K_1, K_4 = O(q^+), \quad K_2 \sim \frac{K_{12}}{p_2^+}, \quad K_3 \sim \frac{K_{34}}{p_3^+} \quad (157)$$

Then it is simple to extract the divergent behavior:

$$\begin{aligned}
& \frac{g^4}{\pi^2} \frac{K_2 K_3 p_4^{+2}}{q^{+2}} \int_{x_2+x_3+x_4 \leq 1} dx_2 dx_3 dx_4 \delta(q^+ - x_{24} p_1^+ - x_{34} p_2^+ - x_4 p_3^+) \frac{1}{H} \\
& \sim \frac{g^4}{\pi^2} \frac{K_{12} K_{34} p_4^+}{q^+ p_1^+ p_2^+ p_3^+} \frac{p_4^+}{(p_1^+ + p_2^+) p_4^2 + p_4^+ (p_1 + p_2)^2} \\
& \left[ \frac{p_1^+ (p_1 + p_2)^2}{(p_1^+ + p_2^+) p_1^2 - p_1^+ (p_1 + p_2)^2} \ln \frac{p_1^+ (p_1 + p_2)^2}{(p_1^+ + p_2^+) p_1^2} + \frac{p_1^+ p_4^2}{p_4^+ p_1^2 + p_1^+ p_4^2} \ln \frac{p_1^+ p_4^2}{(p_1^+ + p_2^+ + p_3^+) p_1^2} \right] \quad (158)
\end{aligned}$$

A similar analysis applies for the region  $q^+ \rightarrow p_1^+ + p_2^+$  with off-shell external legs, which entails  $x_1, x_2, x_4 = O(p_1^+ + p_2^+ - q^+)$ :

$$H \sim x_2 p_2^2 + x_1 (p_1 + p_2)^2 + x_4 p_3^2 = O(q^+) \quad (159)$$

$$K_2, K_3 = O(p_1^+ + p_2^+ - q^+), \quad K_1 \sim \frac{K_{12}}{p_1^+}, \quad K_4 \sim \frac{K_{34}}{p_4^+} \quad (160)$$

$$\begin{aligned}
& \frac{g^4}{\pi^2} \frac{K_1 K_4 p_4^{+2}}{(p_1^+ + p_2^+ - q)^2} \int_{x_2 + x_3 + x_4 \leq 1} dx_2 dx_3 dx_4 \delta(q^+ - x_{24} p_1^+ - x_{34} p_2^+ - x_4 p_3^+) \frac{1}{H} \\
& \sim \frac{g^4}{\pi^2} \frac{K_{12} K_{34} p_4^+}{(p_1^+ + p_2^+ - q^+) p_1^+ p_2^+ p_3^+} \frac{p_3^+}{(p_1^+ + p_2^+) p_3^2 + p_3^+ (p_1 + p_2)^2} \\
& \quad \left[ \frac{p_2^+ (p_1 + p_2)^2}{(p_1^+ + p_2^+) p_2^2 - p_2^+ (p_1 + p_2)^2} \ln \frac{p_2^+ (p_1 + p_2)^2}{(p_1^+ + p_2^+) p_2^2} + \frac{p_2^+ p_3^2}{p_3^+ p_2^2 + p_2^+ p_3^2} \ln \frac{p_2^+ p_3^2}{(p_1^+ + p_2^+ + p_4^+) p_2^2} \right] \quad (161)
\end{aligned}$$

Notice that the on shell limit of these expressions is ambiguous. This happened because they are valid only when  $q^+ \ll p_i^2$ .

To obtain the on-shell scattering of glue by glue, we set  $p_i^2 = 0$  before taking the continuum limit of the  $q^+$  sum. This corresponds to resolving the infra-red divergences, which are essentially symptoms of degenerate state perturbation theory, in the presence of the  $q^+$  cutoff. This is the correct procedure if we commit to  $q^+$  discretization as in defining the theory non-perturbatively, because then, in principle, we should find the exact energy eigenstates of the theory with  $q^+$  discrete, and only at the end take the continuum limit.

Setting all  $p_i^2 = 0$  drastically simplifies  $H$

$$H \rightarrow H_0 = (1 - x_2 - x_3 - x_4) x_3 (p_1 + p_2)^2 + x_2 x_4 (p_2 + p_3)^2 \quad (162)$$

To study the  $q^+$  divergences, we must be careful to not drop terms in  $H_0$  that ensure the convergence of the  $x_i$  integrals. For example in the limit  $q^+ \rightarrow 0$ , even though  $x_2, x_3, x_4 = O(q^+)$ , the integration range includes regions where  $x_3 \ll x_2 x_4$ , so we can't simply drop the second term. Also we must remember that  $H_0$  can be much smaller than  $O(q^+)$ , so we cannot neglect  $x_2 x_4 / H_0^2$  compared to  $1/H_0$  as we could when all legs were off-shell. However it is safe to simplify the first term of  $H_0$ :

$$H_0 \approx x_3 (p_1 + p_2)^2 + x_2 x_4 (p_2 + p_3)^2 \quad \text{for } q^+ \sim 0 \quad (163)$$

It is straightforward to evaluate

$$\int_{x_2 + x_3 + x_4 \leq 1} dx_2 dx_3 dx_4 \delta(q^+ - x_{24} p_1^+ - x_{34} p_2^+ - x_4 p_3^+) \frac{1}{H_0} \sim \frac{q^+}{p_4^+ p_1^+ p_{12}^2} \left[ \ln \frac{q^+ (p_1^+ + p_2^+) p_{23}^2}{-p_4^+ p_1^+ p_{12}^2} - 1 \right] \quad (164)$$

$$\int_{x_2 + x_3 + x_4 \leq 1} dx_2 dx_3 dx_4 \delta(q^+ - x_{24} p_1^+ - x_{34} p_2^+ - x_4 p_3^+) \frac{x_2 x_4}{H_0^2} \sim \frac{q^+}{-p_4^+ p_1^+ p_{12}^2 p_{23}^2} \quad (165)$$

$$\int_{x_2 + x_3 + x_4 \leq 1} dx_2 dx_3 dx_4 \delta(q^+ - x_{24} p_1^+ - x_{34} p_2^+ - x_4 p_3^+) \frac{x_3 x_i}{H_0^2} \sim O(q^{+2}) \quad \text{for } i = 2, 4 \quad (166)$$

Thus, inside the integral we can replace

$$K_1^\wedge K_4^\vee + K_1^\vee K_4^\wedge \rightarrow \frac{x_2 x_4 K_{41}^2}{p_1^+ p_4^+} = -x_2 x_4 p_{23}^2 \quad (167)$$

and putting everything together obtain

$$\Gamma_{\square}^{\wedge\wedge\wedge\vee} \sim \frac{g^4}{\pi^2} \frac{p_4^+ K_{12}^\wedge K_{34}^\wedge}{p_1^+ p_2^+ p_3^+ p_{12}^2} \sum_{q^+} \frac{1}{q^+} \ln \frac{q^+ (p_1^+ + p_2^+) p_{23}^2}{-p_4^+ p_1^+ p_{12}^2} = \Gamma_{0,s}^{\wedge\wedge\wedge\vee} \frac{-g^2}{4\pi^2} \sum_{q^+} \frac{1}{q^+} \ln \frac{q^+ (p_1^+ + p_2^+) p_{23}^2}{-p_4^+ p_1^+ p_{12}^2} \quad (168)$$

where  $\Gamma_{0,s}$  is the direct channel tree amplitude for this scattering process. The analysis of the divergence near  $q^+ = p_1^+ + p_2^+$  analogously yields

$$\begin{aligned}
\Gamma_{\square}^{\wedge\wedge\wedge\vee} & \sim \frac{g^4}{\pi^2} \frac{p_4^+ K_{12}^\wedge K_{34}^\wedge}{p_1^+ p_2^+ p_3^+ p_{12}^2} \sum_{q^+} \frac{1}{p_1^+ + p_2^+ - q^+} \ln \frac{(p_1^+ + p_2^+ - q^+) (p_1^+ + p_2^+) p_{23}^2}{-p_3^+ p_2^+ p_{12}^2} \\
& \sim \Gamma_{0,s}^{\wedge\wedge\wedge\vee} \frac{-g^2}{4\pi^2} \sum_{q^+} \frac{1}{p_1^+ + p_2^+ - q^+} \ln \frac{(p_1^+ + p_2^+ - q^+) (p_1^+ + p_2^+) p_{23}^2}{-p_3^+ p_2^+ p_{12}^2} \quad (169)
\end{aligned}$$



The divergences at  $q^+ = p_1^+, -p_4^+$  are near interior points of the  $q^+$  sum. The leading linear divergence at these points comes from the factors  $(q^+ - p_1^+)^{-2}, (q^+ + p_4^+)^{-2}$  which are multiplied by factors continuous at the singular point. Sub-leading logarithmic divergences can arise when these factors are expanded about the singular points, and the first order corrections are discontinuous there. First order corrections that are continuous at the singular point give rise to factors  $(q^+ - p_1^+)^{-1}, (q^+ + p_4^+)^{-1}$  multiplying smooth functions of  $q^+$  and the continuum limit on  $q^+$  is finite. The leading linear divergence will be exactly canceled by contributions from the quartic triangle diagrams, so here we want to extract the sub-leading divergence that is left after this cancellation. To do this we need to find the discontinuity of the integrand across these singular points (the leading divergence cancels in this discontinuity since it is even).

For definiteness, focus on the divergence at  $p_1^+$ . Discontinuities arise from integrating the delta function factor over one of the  $x_i$ , which leads to different boundaries of integration for the remaining  $x_i$  integrals. For example, integrating over  $x_3$  yields

$$x_3 = \frac{q^+ - x_2 p_1^+ + x_4 p_4^+}{p_1^+ + p_2^+} \quad (170)$$

Then the constraints  $x_3 > 0$  and  $x_2 + x_3 + x_4 < 1$  reduce to the pair of inequalities

$$x_2 p_1^+ - x_4 p_4^+ \leq q^+, \quad x_2 p_2^+ - x_4 p_3^+ \leq p_1^+ + p_2^+ - q^+ \quad (171)$$

which must be simultaneously satisfied. When these equations are equalities they define two negative slope lines in the  $x_2$ - $x_4$  plane. If these lines do not intersect in the first quadrant, the region of integration (always in the first quadrant) is bounded by the one closest to the origin. If they intersect in the first quadrant then the boundary is determined by the part of each line closest to the origin. The intersection point of the two lines is  $(x_2^0, x_4^0)$  where

$$x_2^0 = \frac{q^+ + p_4^+}{p_1^+ + p_4^+} = 1 + \frac{q^+ - p_1^+}{p_1^+ + p_4^+}, \quad x_4^0 = \frac{p_1^+ - q^+}{p_1^+ + p_4^+} = 1 - \frac{q^+ + p_4^+}{p_1^+ + p_4^+} \quad (172)$$

This point is in the first quadrant if either  $-p_4^+ < q^+ < p_1^+$  or  $p_1^+ < q^+ < -p_4^+$ . In either case we see that the character of the boundary changes as  $q^+$  passes through the singular point, causing a discontinuity in behavior of the  $q^+$  summand. For  $q^+$  near  $p_1^+$ , the intersection point is near  $x_2 = 1, x_3, x_4 = 0$ , so the discontinuity and hence the logarithmic divergence comes from this corner of the integration region. Then we can approximate  $H_0$  by

$$\begin{aligned} H_0 &\approx x_4(p_2 + p_3)^2 + x_1 x_3(p_1 + p_2)^2 \\ &\approx x_4(p_2 + p_3)^2 + \left( \frac{q^+ - x_2 p_1^+ + x_4 p_4^+}{p_1^+ + p_2^+} \right) \left( \frac{p_1^+ + p_2^+ - q^+ - x_2 p_2^+ + x_4 p_3^+}{p_1^+ + p_2^+} \right) (p_1 + p_2)^2 \end{aligned} \quad (173)$$

$$K_1, K_2 = O(q^+ - p_1^+), \quad K_3 \approx \frac{K_{23}}{p_3^+}, \quad K_4 \approx \frac{K_{41}}{p_4^+} \quad (174)$$

Since the discontinuity is associated with the boundary of integration it will be useful to write the approximated integrand as a total derivative. To this end, note that

$$\begin{aligned} \mathbf{v} \cdot \nabla H_0 &= (p_4^+ x_4 + p_1^+ x_2)(p_2 + p_3)^2 + \frac{(p_2^+ + p_3^+)(q^+ - x_2 p_1^+ + x_4 p_4^+)}{p_1^+ + p_2^+} (p_1 + p_2)^2 \\ &\approx p_1^+ (p_2 + p_3)^2 + \frac{(p_2^+ + p_3^+)(q^+ - x_2 p_1^+ + x_4 p_4^+)}{p_1^+ + p_2^+} (p_1 + p_2)^2 \\ &\rightarrow p_1^+ (p_2 + p_3)^2 + \frac{(p_2^+ + p_3^+)((1 - x_2)p_1^+ + x_4 p_4^+)}{p_1^+ + p_2^+} (p_1 + p_2)^2 \end{aligned} \quad (175)$$

for  $q^+ = p_1^+$ . Here  $\mathbf{v} \cdot \nabla \equiv p_4^+ \partial_2 + p_1^+ \partial_4$ , and  $\mathbf{v} \cdot \nabla$  applied to the right side of the last line vanishes. Thus we can write any function of  $H_0$  as a total derivative:

$$f'(H_0) = \mathbf{v} \cdot \nabla \left( \frac{f(H_0)}{p_1^+ (p_2 + p_3)^2 + (p_2^+ + p_3^+)(q^+ - x_2 p_1^+ + x_4 p_4^+) (p_1 + p_2)^2 / (p_1^+ + p_2^+)} \right) \quad (176)$$

Then we can write the discontinuity of the amplitude across  $q^+ = p_1^+$  as

$$\begin{aligned}\Gamma^{\wedge\wedge\wedge\wedge} &\sim \frac{g^4}{\pi^2} \sum_{q^+} \frac{K_{23}^\wedge K_{41}^\wedge p_4^{+2}}{p_3^+ p_4^+ (q^+ - p_1^+)^2} \frac{1}{2} \text{Disc} \int_{\mathcal{R}} \frac{dx_2 dx_4}{p_1^+ + p_2^+} \left( \frac{K_1^\vee K_2^\wedge + K_1^\wedge K_2^\vee}{H_0^2} + \frac{1}{H_0} \right) \\ &\sim \frac{g^4}{\pi^2} \sum_{q^+} \frac{K_{23}^\wedge K_{41}^\wedge p_4^{+2}}{p_3^+ p_4^+ (q^+ - p_1^+)^2} \frac{1}{2} \text{Disc} \oint_{\partial\mathcal{R}} d\ell \mathbf{v} \cdot \hat{\mathbf{n}} \left( \frac{\ln H_0 - K_1 \cdot K_2 H_0^{-1}}{p_1^+ (p_1^+ + p_2^+) (p_2 + p_3)^2} \right)\end{aligned}$$

where we have simplified the denominator in the last line by dropping terms which vanish for  $q^+ = p_1^+$ ,  $x_4 = 0$ ,  $x_2 = 1$ , since the discontinuity only receives contributions near this point. For definiteness take  $p_1^+ + p_4^+ < 0$ . Then for  $q^+ < p_1^+$  the line  $x_2 p_1^+ - x_4 p_4^+ = q^+$  lies closest to the origin and forms a part of  $\partial\mathcal{R}$ . On this part  $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$  so it does not contribute. The axes  $x_4 = 0$  and  $x_2 = 0$  form the rest of the boundary, but only the neighborhood of the point  $x_2 = 1, x_4 = 0$  contributes to the discontinuity. On the  $x_4 = 0$  axis,  $\mathbf{v} \cdot \hat{\mathbf{n}} = -p_1^+$  and  $H_0$  reduces to

$$H_0 \rightarrow \left( \frac{q^+ - x_2 p_1^+}{p_1^+ + p_2^+} \right) \left( \frac{p_1^+ + p_2^+ - q^+ - x_2 p_2^+}{p_1^+ + p_2^+} \right) (p_1 + p_2)^2 \quad (177)$$

Furthermore, in this region  $K_1 \cdot K_2 H_0^{-1} \rightarrow -1$  so the relevant integral is just

$$\begin{aligned}-p_1^+ \int_{1-\epsilon}^{q^+/p_1^+} dx_2 \frac{\ln H_0 + 1}{p_1^+ (p_1^+ + p_2^+)} = \\ -\frac{q^+}{p_1^+ (p_1^+ + p_2^+)} \left( 1 + \ln \frac{(p_1 + p_2)^2}{(p_1^+ + p_2^+)^2} \right) + \frac{|p_1^+ - q^+|}{p_1^+ p_2^+} \left( \ln \frac{(p_1^+ + p_2^+) |p_1^+ - q^+|}{p_1^+} - 1 \right) + f(1 - \epsilon) \quad (178)\end{aligned}$$

In contrast, for  $q^+ > p_1^+$  the contributing boundary in the neighborhood of the point  $x_2 = 1, x_4 = 0$  includes not only a segment on the  $x_4 = 0$  axis up to  $x_2 = (p_1^+ + p_2^+ - q^+)/p_2^+$ , but also the segment of the line  $x_2 p_2^+ - x_4 p_3^+ = p_1^+ + p_2^+ - q^+$  between the  $x_4 = 0$  axis and its intersection with the line  $x_2 p_1^+ - x_4 p_4^+ = q^+$ , which then takes over as boundary, but contributes nothing because  $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$  on it. The first part contributes

$$\begin{aligned}-p_1^+ \int_{1-\epsilon}^{(p_1^+ + p_2^+ - q^+)/p_2^+} dx_2 \left( \frac{\ln H_0 + 1}{p_1^+ (p_1^+ + p_2^+)} \right) = \\ -\frac{p_1^+ + p_2^+ - q^+}{p_2^+ (p_1^+ + p_2^+)} \left( 1 + \ln \frac{(p_1 + p_2)^2}{(p_1^+ + p_2^+)^2} \right) + \frac{|p_1^+ - q^+|}{p_1^+ p_2^+} \left( \ln \frac{(p_1^+ + p_2^+) |p_1^+ - q^+|}{p_2^+} - 1 \right) + f(1 - \epsilon) \quad (179)\end{aligned}$$

With a little rearrangement it is straightforward to show that these apparently different expressions for  $q^+ < p_1^+$  and  $q^+ > p_1^+$  can be written in the unified form

$$\frac{|p_1^+ - q^+|}{2p_1^+ p_2^+} \left( \ln \frac{(p_1 + p_2)^2 |p_1^+ - q^+|^2}{p_1^+ p_2^+} - 1 \right) + g(q^+) \quad (180)$$

where  $g$  and its first derivative are continuous at  $q^+ = p_1^+$ . To this we must add the contribution (when  $q^+ > p_1^+$ ) of the segment of the line  $x_2 p_2^+ - x_4 p_3^+ = p_1^+ + p_2^+ - q^+$ . On this line  $H_0 = x_4 (p_2 + p_3)^2 = (p_1^+ + p_2^+ - q^+ - x_2 p_2^+) / (-p_3^+)$ ,  $K_1 \cdot k_2 \approx 0$  and  $d\ell \mathbf{v} \cdot \hat{\mathbf{n}} = dx_2 (p_1^+ - p_2^+ p_4^+ / p_3^+)$  and the relevant integral is

$$\frac{1}{p_1^+} \int_{x_2^0}^{(p_1^+ + p_2^+ - q^+)/p_2^+} dx_2 (p_1^+ - p_2^+ p_4^+ / p_3^+) \ln H_0 = \frac{(q^+ - p_1^+)}{p_1^+ p_2^+} \left( \ln \frac{p_2^+ + p_3^+}{(q^+ - p_1^+) (p_2 + p_3)^2} + 1 \right) \quad (181)$$

which contributes only for  $q^+ > p_1^+$ . But we can write the same singular behavior as a contribution for both  $q^+, p_1^+$  and  $q^+ > p_1^+$  by enclosing  $q^+ - p_1^+$  in absolute value signs and multiplying by  $1/2$ . Then the

complete contribution to the singular behavior near  $q^+ = p_1^+$  to the box diagram after cancellation of the linear divergence with the quartic triangle diagram can be written:

$$\Gamma_{\square}^{\wedge\wedge\wedge\vee} \sim \Gamma_{0,t}^{\wedge\wedge\wedge\vee} \frac{-g^2}{4\pi^2} \sum_{q^+ \neq p_1^+} \frac{1}{2|p_1^+ - q^+|} \ln \frac{|p_1^+ - q^+|(p_2^+ + p_3^+)p_{12}^2}{p_1^+ p_2^+ p_{23}^2} \quad (182)$$

We see that after the dust has settled, even though the analysis is very different between end and interior points the final result for the singular behavior is exactly analogous. Thus we can immediately write the result for the singular behavior near  $q^+ = -p_4^+$ :

$$\Gamma_{\square}^{\wedge\wedge\wedge\vee} \sim \Gamma_{0,t}^{\wedge\wedge\wedge\vee} \frac{-g^2}{4\pi^2} \sum_{q^+ \neq -p_4^+} \frac{1}{2|q^+ + p_4^+|} \ln \frac{|q^+ + p_4^+|(p_2^+ + p_3^+)p_{12}^2}{p_3^+ p_4^+ p_{23}^2} \quad (183)$$

## 6.2 Quartic Triangle Diagrams

The quartic triangle diagrams are labeled by the pair of legs entering the quartic vertex. We first evaluate

$$\begin{aligned} \Gamma_{12}^{\wedge\wedge\wedge\vee} &= -2g^4 \sum_{q^+} \int dT_1 dT_3 dT_4 \delta((T_1 + T_{34})q^+ - T_{34}(p_1^+ + p_2^+) - T_4 p_3^+) \int \frac{d^2 q}{(2\pi)^3} \\ &\quad \exp \left\{ -\frac{\delta(\mathbf{k}_0 T_1 + \mathbf{k}_2 T_3 + \mathbf{k}_3 T_4)^2}{T_{14}(T_{14} + \delta)} - (T_{14} + \delta) \mathbf{q}^2 \right\} \\ &\quad \exp \left\{ -\frac{T_1 T_3 (p_1 + p_2)^2 + T_1 T_4 p_4^2 + T_3 T_4 p_3^2}{T_{14}} \right\} \\ &\quad \frac{p_4^{+2}}{q^+(p_1^+ + p_2^+ - q^+)} \left[ \frac{(q^+ + p_1^+)(p_1^+ + 2p_2^+ - q^+)}{(q^+ - p_1^+)^2} + 1 \right] (q + K_3)^\wedge (q + K_4)^\wedge \end{aligned} \quad (184)$$

$$\begin{aligned} &\sim -\frac{g^4}{4\pi^2} \sum_{q^+} \int \frac{dT_1 dT_3 dT_4}{T_1 + T_{34}} \delta((T_1 + T_{34})q^+ - T_{34}(p_1^+ + p_2^+) - T_4 p_3^+) \\ &\quad \exp \left\{ -\frac{T_1 T_3 (p_1 + p_2)^2 + T_1 T_4 p_4^2 + T_3 T_4 p_3^2}{T_{14}} \right\} \\ &\quad \frac{p_4^{+2}}{q^+(p_1^+ + p_2^+ - q^+)} \left[ \frac{(q^+ + p_1^+)(p_1^+ + 2p_2^+ - q^+)}{(q^+ - p_1^+)^2} + 1 \right] \frac{T_1 T_3 K_{34}^{\wedge 2}}{p_3^+ p_4^+ T_{14}^2} \\ &= -\frac{g^4}{4\pi^2} \sum_{q^+} \int_{x_3 + x_4 \leq 1} dx_3 dx_4 \delta(q^+ - x_{34}(p_1^+ + p_2^+) - x_4 p_3^+) \\ &\quad \frac{p_4^{+2}}{q^+(p_1^+ + p_2^+ - q^+)} \left[ \frac{(q^+ + p_1^+)(p_1^+ + 2p_2^+ - q^+)}{(q^+ - p_1^+)^2} + 1 \right] \frac{(1 - x_3 - x_4)x_3 K_{34}^{\wedge 2}}{p_3^+ p_4^+ H} \end{aligned} \quad (185)$$

with

$$H = (1 - x_3 - x_4)x_3(p_1 + p_2)^2 + (1 - x_3 - x_4)x_4 p_4^2 + x_3 x_4 p_3^2 \quad (186)$$

Using the delta function to eliminate  $x_3 = (q^+ + x_4 p_4^+)/ (p_1^+ + p_2^+)$ , we find that the upper limit on  $x_4$  is  $q^+ / (-p_4^+)$  if  $q^+ < -p_4^+$  and  $(p_1^+ + p_2^+ - q^+) / (-p_3^+)$  if  $q^+ > -p_4^+$ . Thus the only  $q^+$  singularity is at  $q^+ = p_1^+$ . Putting the external lines on shell we find

$$\begin{aligned} \Gamma_{12}^{\wedge\wedge\wedge\vee} &\rightarrow \frac{g^4}{4\pi^2} \frac{K_{34}^{\wedge 2}}{p_3^+ p_4^+ (p_1^+ + p_2^+) (p_1 + p_2)^2} \left\{ \sum_{q^+ < -p_4^+} \frac{2p_4^+}{(p_1^+ + p_2^+ - q^+)} \left[ \frac{p_1^+ (p_1^+ + p_2^+ - q^+) + p_2^+ q^+}{(q^+ - p_1^+)^2} \right] \right. \\ &\quad \left. + \sum_{q^+ > -p_4^+} \frac{2p_4^{+2}}{p_3^+ q^+} \left[ \frac{p_1^+ (p_1^+ + p_2^+ - q^+) + p_2^+ q^+}{(q^+ - p_1^+)^2} \right] \right\} \end{aligned} \quad (187)$$

The continuum limit of the  $q^+$  sum is singular only due to the factors  $(q^+ - p_1^+)^{-2}$ . The leading linear divergence is necessary to cancel the corresponding divergence in the box diagram. As long as  $p_1^+ + p_4^+ \neq 0$  the potential logarithmic sub-divergence is absent because the  $q^+$  discretization implies a principle value prescription. Next

$$\begin{aligned} \Gamma_{23}^{\wedge\wedge\wedge\vee} &= -2g^4 \sum_{q^+} \int dT_1 dT_2 dT_4 \delta((T_{12} + T_4)q^+ - T_2 p_1^+ + T_4 p_4^+) \int \frac{d^2 q}{(2\pi)^3} \\ &\quad \exp \left\{ -\frac{\delta(\mathbf{k}_0 T_1 + \mathbf{k}_1 T_2 + \mathbf{k}_3 T_4)^2}{T_{14}(T_{14} + \delta)} - (T_{14} + \delta) \mathbf{q}^2 \right\} \\ &\quad \exp \left\{ -\frac{T_2 T_4 (p_2 + p_3)^2 + T_1 T_4 p_4^2 + T_1 T_2 p_1^2}{T_{14}} \right\} \\ &\quad \frac{p_4^{+2}}{(q^+ + p_4^+)(p_1^+ - q^+)} \left[ \frac{(p_2^+ + q^+ - p_1^+)(p_3^+ - p_4^+ - q^+)}{(q^+ - p_1^+ - p_2^+)^2} + 1 \right] (q + K_1)^\wedge (q + K_4)^\wedge \quad (188) \end{aligned}$$

$$\begin{aligned} &\sim -\frac{g^4}{4\pi^2} \sum_{q^+} \int_{x_2 + x_4 \leq 1} dx_2 dx_4 \delta(q^+ - x_2 p_1^+ + x_4 p_4^+) \\ &\quad \frac{p_4^{+2}}{(q^+ + p_4^+)(p_1^+ - q^+)} \left[ \frac{(p_2^+ + q^+ - p_1^+)(p_3^+ - p_4^+ - q^+)}{(q^+ - p_1^+ - p_2^+)^2} + 1 \right] \frac{x_2 x_4 K_{41}^{\wedge 2}}{p_1^+ p_4^+ H} \quad (189) \end{aligned}$$

with

$$H = x_2 x_4 (p_2 + p_3)^2 + (1 - x_2 - x_4) x_4 p_4^2 + (1 - x_2 - x_4) x_2 p_1^2 \quad (190)$$

The delta function implies that  $q^+ < \max(p_1^+, |p_4^+|)$ . In the case  $p_1^+ + p_4^+ > 0$ , we use the delta function to eliminate  $x_2 = (q^+ + x_4 p_4^+)/p_1^+$ , then the lower limit on  $x_4$  is 0, and we find that the upper limit on  $x_4$  is  $q^+ / (-p_4^+)$  if  $q^+ < -p_4^+$  and  $(p_1^+ - q^+) / (p_1^+ + p_4^+)$  if  $p_1^+ > q^+ > -p_4^+$ . Thus the only  $q^+$  singularity is at  $q^+ = -p_4^+$ , since  $q^+$  is prevented from approaching  $p_1^+ + p_2^+$ . In the other case,  $p_1^+ + p_4^+ < 0$ , we eliminate  $x_4 = (q^+ - x_2 p_1^+) / |p_4^+|$ , and find the upper limit on  $x_2$  to be  $q^+ / p_1^+$  for  $q^+ < p_1^+$  and  $(|p_4^+| - q^+) / |p_1^+ + p_4^+|$  for  $p_1^+ < q^+ < |p_4^+|$ . Putting the external lines on shell and assuming  $p_1^+ + p_4^+ > 0$  we find

$$\begin{aligned} \Gamma_{23}^{\wedge\wedge\wedge\vee} &\rightarrow -\frac{g^4}{4\pi^2} \frac{K_{41}^{\wedge 2}}{p_1^+ p_4^+ (p_2 + p_3)^2} \left\{ \sum_{q^+ < -p_4^+} \frac{-2|p_4^+| q^+}{p_1^+ (q^+ + p_4^+)(p_1^+ - q^+)} \left[ \frac{p_2^+ (p_4^+ + q^+) + p_3^+ (p_1^+ - q^+)}{(q^+ - p_1^+ - p_2^+)^2} \right] \right. \\ &\quad \left. + \sum_{p_1^+ > q^+ > -p_4^+} \frac{-2p_4^{+2}}{p_1^+ (q^+ + p_4^+)(p_1^+ + p_4^+)} \left[ \frac{p_2^+ (p_4^+ + q^+) + p_3^+ (p_1^+ - q^+)}{(q^+ - p_1^+ - p_2^+)^2} \right] \right\} \quad (191) \end{aligned}$$

The continuum limit of the  $q^+$  sum is finite because it dictates a principal value prescription for the singularity at  $q^+ = -p_4^+$ . The same conclusion applies to the case  $p_1^+ + p_4^+ < 0$ .

The remaining two cases of this class of diagrams yield lengthier expressions since more than one spin flow is possible.

$$\begin{aligned} \Gamma_{34}^{\wedge\wedge\wedge\vee} &= -2g^4 \sum_{q^+} \int dT_1 dT_2 dT_3 \delta(T_{13} q^+ - T_{23} p_1^+ - T_3 p_2^+) \int \frac{d^2 q}{(2\pi)^3} \\ &\quad \exp \left\{ -\frac{\delta(\mathbf{k}_0 T_1 + \mathbf{k}_1 T_2 + \mathbf{k}_2 T_3)^2}{T_{13}(T_{13} + \delta)} - (T_{13} + \delta) \mathbf{q}^2 \right\} \\ &\quad \exp \left\{ -\frac{T_1 T_3 (p_1 + p_2)^2 + T_1 T_2 p_1^2 + T_2 T_3 p_3^2}{T_{13}} \right\} \\ &\quad \left[ 2 \frac{q^+}{p_1^+ + p_2^+ - q^+} \left( \frac{q^+ p_4^+ + (p_1^+ + p_2^+ - q^+) p_3^+}{(p_1^+ + p_2^+)^2} + \frac{q^+ (p_1^+ + p_2^+ - q^+) + p_3^+ p_4^+}{(q^+ + p_4^+)^2} \right) \right] \end{aligned}$$

$$\left. -2 \frac{(p_1^+ + p_2^+ - q^+)}{q^+} \frac{(q^+ p_3^+ + (p_1^+ + p_2^+ - q^+) p_4^+)}{(p_1^+ + p_2^+)^2} \right] (q + K_1)^\wedge (q + K_2)^\wedge \quad (192)$$

$$\begin{aligned} \sim & -\frac{g^4}{4\pi^2} \sum_{q^+} \int_{x_2+x_3 \leq 1} dx_2 dx_3 \delta(q^+ - x_{23} p_1^+ - x_3 p_2^+) \\ & \left[ 2 \frac{q^+}{p_1^+ + p_2^+ - q^+} \left( \frac{q^+ p_4^+ + (p_1^+ + p_2^+ - q^+) p_3^+}{(p_1^+ + p_2^+)^2} + \frac{q^+ (p_1^+ + p_2^+ - q^+) + p_3^+ p_4^+}{(q^+ + p_4^+)^2} \right) \right. \\ & \left. - 2 \frac{(p_1^+ + p_2^+ - q^+)}{q^+} \frac{(q^+ p_3^+ + (p_1^+ + p_2^+ - q^+) p_4^+)}{(p_1^+ + p_2^+)^2} \right] \frac{(1 - x_2 - x_3) x_3 K_{12}^{\wedge 2}}{p_1^+ p_2^+ H} \end{aligned} \quad (193)$$

with

$$H = (1 - x_2 - x_3) x_3 (p_1 + p_2)^2 + (1 - x_2 - x_3) x_2 p_1^2 + x_2 x_3 p_2^2 \quad (194)$$

Using the delta function to eliminate  $x_3 = (q^+ - x_2 p_1^+) / (p_1^+ + p_2^+)$ , we find that the upper limit on  $x_2$  is  $q^+ / p_1^+$  for  $q^+ < p_1^+$  and  $(p_1^+ + p_2^+ - q^+) / p_2^+$  for  $q^+ > p_1^+$ . Putting the external legs on shell then gives

$$\begin{aligned} \Gamma_{34}^{\wedge\wedge\wedge\vee} \rightarrow & -\frac{g^4}{4\pi^2} \frac{K_{12}^{\wedge 2}}{p_1^+ p_2^+ (p_1^+ + p_2^+) (p_1 + p_2)^2} \left\{ \right. \\ & \sum_{q^+ < p_1^+} \left[ 2 \frac{q^{+2}}{p_1^+ (p_1^+ + p_2^+ - q^+)} \left( \frac{q^+ p_4^+ + (p_1^+ + p_2^+ - q^+) p_3^+}{(p_1^+ + p_2^+)^2} + \frac{q^+ (p_1^+ + p_2^+ - q^+) + p_3^+ p_4^+}{(q^+ + p_4^+)^2} \right) \right. \\ & \left. - 2 \frac{(p_1^+ + p_2^+ - q^+)}{p_1^+} \frac{(q^+ p_3^+ + (p_1^+ + p_2^+ - q^+) p_4^+)}{(p_1^+ + p_2^+)^2} \right] \\ & \sum_{q^+ > p_1^+} \left[ 2 \frac{q^+}{p_2^+} \left( \frac{q^+ p_4^+ + (p_1^+ + p_2^+ - q^+) p_3^+}{(p_1^+ + p_2^+)^2} + \frac{q^+ (p_1^+ + p_2^+ - q^+) + p_3^+ p_4^+}{(q^+ + p_4^+)^2} \right) \right. \\ & \left. \left. - 2 \frac{(p_1^+ + p_2^+ - q^+)^2}{p_2^+ q^+} \frac{(q^+ p_3^+ + (p_1^+ + p_2^+ - q^+) p_4^+)}{(p_1^+ + p_2^+)^2} \right] \right\} \end{aligned} \quad (195)$$

The only  $q^+$  divergence here is due to the factor  $(q^+ + p_4^+)^{-2}$  and as in the 12 case, the leading divergence cancels a corresponding divergence in the box diagram and there is no sub-leading divergence because of the principal value prescription.

The final case is

$$\begin{aligned} \Gamma_{41}^{\wedge\wedge\wedge\vee} = & -2g^4 \sum_{q^+} \int dT_2 dT_3 dT_4 \delta(T_{24} q^+ - T_{24} p_1^+ - T_{34} p_2^+ - T_4 p_3^+) \int \frac{d^2 q}{(2\pi)^3} \\ & \exp \left\{ -\frac{\delta(\mathbf{k}_1 T_2 + \mathbf{k}_2 T_3 + \mathbf{k}_3 T_4)^2}{T_{24}(T_{24} + \delta)} - (T_{24} + \delta) \mathbf{q}^2 \right\} \\ & \exp \left\{ -\frac{T_2 T_4 (p_2 + p_3)^2 + T_2 T_3 p_2^2 + T_3 T_4 p_3^2}{T_{24}} \right\} \\ & \left[ 2 \frac{q^+ + p_4^+}{p_1^+ - q^+} \left( \frac{(q^+ - p_1^+) p_1^+ - (q^+ + p_4^+) p_4^+}{(p_1^+ + p_4^+)^2} + \frac{(q^+ + p_4^+) (p_1^+ - q^+) + p_1^+ p_4^+}{q^{+2}} \right) \right. \\ & \left. - 2 \frac{p_1^+ - q^+}{q^+ + p_4^+} \frac{(q^+ - p_1^+) p_4^+ - (q^+ + p_4^+) p_1^+}{(p_1^+ + p_4^+)^2} \right] (q + K_2)^\wedge (q + K_3)^\wedge \quad (196) \\ \sim & -\frac{g^4}{4\pi^2} \sum_{q^+} \int_{x_2+x_4 \leq 1} dx_2 dx_4 \delta(q^+ - p_1^+ - (1 - x_2) p_2^+ - x_4 p_3^+) \\ & \left[ 2 \frac{q^+ + p_4^+}{p_1^+ - q^+} \left( \frac{(q^+ - p_1^+) p_1^+ - (q^+ + p_4^+) p_4^+}{(p_1^+ + p_4^+)^2} + \frac{(q^+ + p_4^+) (p_1^+ - q^+) + p_1^+ p_4^+}{q^{+2}} \right) \right. \end{aligned}$$

$$-2 \frac{p_1^+ - q^+}{q^+ + p_4^+} \frac{(q^+ - p_1^+)p_4^+ - (q^+ + p_4^+)p_1^+}{(p_1^+ + p_4^+)^2} \Big] \frac{x_2 x_4 K_{23}^{\wedge 2}}{p_2^+ p_3^+ H} \quad (197)$$

with

$$H = x_2 x_4 (p_2 + p_3)^2 + (1 - x_2 - x_4) x_2 p_2^2 + (1 - x_2 - x_4) x_4 p_3^2 \quad (198)$$

The  $x_2, x_4$  integration in this case closely parallels the procedure in the 23 case with the substitutions  $q^+ \rightarrow p_1^+ + p_2^+ - q^+$ ,  $p_1^+ \rightarrow p_2^+$ ,  $p_4^+ \rightarrow p_3^+$ . Then the cases  $p_2^+ + p_3^+ < 0$ ,  $p_2^+ + p_3^+ > 0$  are handled separately. Just as in that case we find here that the continuum limit of the  $q^+$  sum is finite. For the case  $p_{23}^+ < 0$ , we find

$$\begin{aligned} \Gamma_{41}^{\wedge \wedge \wedge \vee} = & -\frac{g^4}{4\pi^2} \frac{2K_{23}^{\wedge 2}}{p_2^+ p_3^+ p_{12}^2} \left\{ \sum_{-p_4^+ < q^+ < p_1^+} \frac{q^+ + p_4^+}{p_{14}^+} \left[ \frac{q^+ + p_4^+}{p_1^+ - q^+} \right. \right. \\ & \left( \frac{(q^+ - p_1^+)p_1^+ - (q^+ + p_4^+)p_4^+}{(p_1^+ + p_4^+)^2} + \frac{(q^+ + p_4^+)(p_1^+ - q^+) + p_1^+ p_4^+}{q^{+2}} \right) - \frac{p_1^+ - q^+}{q^+ + p_4^+} \frac{(q^+ - p_1^+)p_4^+ - (q^+ + p_4^+)p_1^+}{(p_1^+ + p_4^+)^2} \Big] \\ & + \sum_{q^+ > p_1^+} \frac{p_{12}^+ - q^+}{p_2^+} \left[ \frac{q^+ + p_4^+}{p_1^+ - q^+} \left( \frac{(q^+ - p_1^+)p_1^+ - (q^+ + p_4^+)p_4^+}{(p_1^+ + p_4^+)^2} + \frac{(q^+ + p_4^+)(p_1^+ - q^+) + p_1^+ p_4^+}{q^{+2}} \right) \right. \\ & \left. \left. - \frac{p_1^+ - q^+}{q^+ + p_4^+} \frac{(q^+ - p_1^+)p_4^+ - (q^+ + p_4^+)p_1^+}{(p_1^+ + p_4^+)^2} \right] \right\} \quad (199) \end{aligned}$$

To summarize this section, the  $q^+$  divergences of the quartic triangle diagrams are precisely what is needed to remove the linear divergences in the box diagrams and they contribute no sub-leading logarithmic divergences.

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## A Some Useful Integrals

In the evaluation of the on-shell triangle diagram, we encounter integrals of the form

$$\int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)p_1^+ - yp_2^+) \mathcal{I} \quad (200)$$

where the integrand is a linear function of  $x$  times a linear function of  $\ln xy$ ,  $\ln x(1-x-y)$ , or  $\ln y(1-x-y)$ . By  $p^+$  conservation, two of the momenta  $p_{1,2,3}^+$  have one sign and the third has the opposite sign. In this section we label momenta so that  $p_1^+ > 0$  and  $p_3^+ < 0$ . If  $p_2^+$  is positive do the above integral in its displayed form. If  $p_2^+$  is negative rewrite the argument of the delta function in terms of  $p_2^+ = -|p_2^+|$  and  $p_3^+ = -|p_3^+|$ , and rename  $x \leftrightarrow y$ , which brings the integral to the form

$$\int_{x+y \leq 1} dx dy \delta(q^+ - (x+y)|p_3^+| - y|p_2^+|) \mathcal{I} \quad (201)$$

which reduces it to the first form, with  $|p_3^+|$  in the role of  $p_1^+$  and  $|p_2^+|$  in the role of  $p_2^+$ . Thus, without loss of generality we can stipulate that  $p_1^+, p_2^+ > 0$ . Then we do the  $y$  integral which sets  $y = (q^+ - xp_1^+)/p_{12}^+$ , and also sets the range of the  $x$  integral  $0 < x < x_m$  where  $x_m = q^+/p_1^+$  for  $q^+ < p_1^+$  and  $x_m = (p_{12}^+ - q^+)/p_2^+$  for  $q^+ > p_1^+$ . Then the following  $x$  integrals are needed:

$$\int dx \ln(xy) = \left( x_m - \frac{q^+}{p_1^+} \right) \ln \frac{q^+ - x_m p_1^+}{p_{12}^+} + \frac{q^+}{p_1^+} \ln \frac{q^+}{p_{12}^+} + x_m \ln x_m - 2x_m \quad (202)$$

$$= \begin{cases} \frac{q^+}{p_1^+} \left( \ln \frac{q^+}{p_{12}^+} + \ln \frac{q^+}{p_1^+} - 2 \right) & \text{for } q^+ < p_1^+ \\ \frac{p_{12}^+(p_1^+ - q^+)}{p_1^+ p_2^+} \ln \frac{q^+ - p_1^+}{p_2^+} + \frac{q^+}{p_1^+} \ln \frac{q^+}{p_{12}^+} \\ \quad + \frac{p_{12}^+ - q^+}{p_2^+} \left( \ln \frac{p_{12}^+ - q^+}{p_2^+} - 2 \right) & \text{for } q^+ > p_1^+ \end{cases} \quad (203)$$

$$\begin{aligned} \int dx \, x \ln(xy) &= \frac{1}{2} \left( x_m^2 - \frac{q^{+2}}{p_1^{+2}} \right) \ln \frac{q^+ - x_m p_1^+}{p_{12}^+} + \frac{q^{+2}}{2p_1^{+2}} \ln \frac{q^+}{p_{12}^+} + \frac{x_m^2}{2} \ln x_m - \frac{x_m^2 p_1^+ + q^+ x_m}{2p_1^+} \\ &= \begin{cases} \frac{q^{+2}}{2p_1^{+2}} \left( \ln \frac{q^+}{p_{12}^+} + \ln \frac{q^+}{p_1^+} - 2 \right) & \text{for } q^+ < p_1^+ \\ \frac{p_{12}^+(p_1^+ - q^+)}{2p_1^+ p_2^+} \left[ \frac{q^+}{p_1^+} + \frac{p_{12}^+ - q^+}{p_2^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} + \frac{q^{+2}}{2p_1^{+2}} \ln \frac{q^+}{p_{12}^+} \\ \quad + \frac{(p_{12}^+ - q^+)^2}{2p_2^{+2}} \left( \ln \frac{p_{12}^+ - q^+}{p_2^+} - 1 \right) - \frac{q^+(p_{12}^+ - q^+)}{2p_1^+ p_2^+} & \text{for } q^+ > p_1^+ \end{cases} \quad (204) \end{aligned}$$

$$\int dx \, \ln x(1-x-y) = \begin{cases} \frac{q^+}{p_1^+} \left( \ln \frac{q^+}{p_1^+} - 2 \right) + \frac{p_{12}^+ - q^+}{p_2^+} \ln \frac{p_{12}^+ - q^+}{p_{12}^+} \\ \quad - \frac{p_{12}^+(p_1^+ - q^+)}{p_1^+ p_2^+} \ln \frac{p_1^+ - q^+}{p_1^+} & \text{for } q^+ < p_1^+ \\ \frac{p_{12}^+ - q^+}{p_2^+} \left( \ln \frac{p_{12}^+ - q^+}{p_{12}^+} + \ln \frac{p_{12}^+ - q^+}{p_2^+} - 2 \right) & \text{for } q^+ > p_1^+ \end{cases} \quad (205)$$

$$\int dx \, x \ln(x(1-x-y)) = \begin{cases} \frac{q^{+2}}{2p_1^{+2}} \left( \ln \frac{q^+}{p_1^+} - 1 \right) + \frac{(p_{12}^+ - q^+)^2}{2p_2^{+2}} \ln \frac{p_{12}^+ - q^+}{p_{12}^+} - \frac{q^+(p_{12}^+ - q^+)}{2p_1^+ p_2^+} \\ \quad - \frac{p_{12}^+(p_1^+ - q^+)}{2p_1^+ p_2^+} \left[ \frac{q^+}{p_1^+} + \frac{p_{12}^+ - q^+}{p_2^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} & \text{for } q^+ < p_1^+ \\ \frac{(p_{12}^+ - q^+)^2}{2p_2^{+2}} \left( \ln \frac{p_{12}^+ - q^+}{p_{12}^+} + \ln \frac{p_{12}^+ - q^+}{p_2^+} - 2 \right) & \text{for } q^+ > p_1^+ \end{cases} \quad (206)$$

$$\int dx \, \ln y(1-x-y) = \begin{cases} \frac{q^+}{p_1^+} \left( \ln \frac{q^+}{p_{12}^+} - 2 \right) + \frac{p_{12}^+ - q^+}{p_2^+} \ln \frac{p_{12}^+ - q^+}{p_{12}^+} \\ \quad - \frac{p_{12}^+(p_1^+ - q^+)}{p_1^+ p_2^+} \ln \frac{p_1^+ - q^+}{p_1^+} & \text{for } q^+ < p_1^+ \\ \frac{q^+}{p_1^+} \ln \frac{q^+}{p_{12}^+} + \frac{p_{12}^+ - q^+}{p_2^+} \left( \ln \frac{p_{12}^+ - q^+}{p_{12}^+} - 2 \right) \\ \quad - \frac{p_{12}^+(q^+ - p_1^+)}{p_1^+ p_2^+} \ln \frac{q^+ - p_1^+}{p_2^+} & \text{for } q^+ > p_1^+ \end{cases} \quad (207)$$

$$\int dx \, x \ln(y(1-x-y)) = \begin{cases} \frac{q^{+2}}{2p_1^{+2}} \left( \ln \frac{q^+}{p_{12}^+} - 2 \right) + \frac{(p_{12}^+ - q^+)^2}{2p_2^{+2}} \ln \frac{p_{12}^+ - q^+}{p_{12}^+} - \frac{q^+(p_{12}^+ - q^+)}{2p_1^+ p_2^+} \\ - \frac{p_{12}^+(p_1^+ - q^+)}{2p_1^+ p_2^+} \left[ \frac{q^+}{p_1^+} + \frac{p_{12}^+ - q^+}{p_2^+} \right] \ln \frac{p_1^+ - q^+}{p_1^+} & \text{for } q^+ < p_1^+ \\ \frac{q^{+2}}{2p_1^{+2}} \ln \frac{q^+}{p_{12}^+} + \frac{(p_{12}^+ - q^+)^2}{2p_2^{+2}} \left( \ln \frac{p_{12}^+ - q^+}{p_{12}^+} - 2 \right) - \frac{q^+(p_{12}^+ - q^+)}{2p_1^+ p_2^+} \\ - \frac{p_{12}^+(q^+ - p_1^+)}{2p_1^+ p_2^+} \left[ \frac{q^+}{p_1^+} + \frac{p_{12}^+ - q^+}{p_2^+} \right] \ln \frac{q^+ - p_1^+}{p_2^+} & \text{for } q^+ > p_1^+ \end{cases} \quad (208)$$

## B Other Spin configurations

In the text we analyzed the one-loop three gluon vertex with spin configuration  $\wedge \wedge \vee$  in complete detail. In this appendix we briefly summarize the situation for the other spin configurations. We maintain the choice of two incoming particles and one outgoing particle, so  $p_1^+, p_2^+ > 0$  and  $p_3^+ < 0$ .

### B.1 The remaining triangle diagrams

For the triangle diagram, the other spin configurations are obtained by modifying the  $A_i$  appearing in Eq. (27) as described in the attached footnote, where we called them  $A_i^j$  where  $j$  labels the leg with the down spin (so  $A_i^3 \equiv A_i$ ). It is straightforward to work out the consequences of these modifications.

First, the surface terms after the integration by parts become:

$$\begin{aligned} & \text{Surface Terms}^{\vee \wedge \wedge} \\ &= -\frac{g^3}{4\pi^2} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) \right. \\ &+ \sum_{q^+ < p_1^+} \left[ \frac{1}{p_{12}^+ - q^+} + \frac{2}{q^+} + \frac{1}{p_1^+ - q^+} \right] + \sum_{q^+ > p_1^+} \left[ \frac{2}{p_{12}^+ - q^+} + \frac{1}{q^+} + \frac{1}{q^+ - p_1^+} \right] \\ &+ \sum_{q^+ < p_1^+} \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^2 (p_{12}^+)^2 (p_1^+ - q^+)}{(p_1 + p_2)^2 p_1^{+2} (p_{12}^+ - q^+)} \\ &+ \sum_{q^+ > p_1^+} \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_2^2 (p_1^+ + p_2^+)^2 (q^+ - p_1^+)}{(p_1 + p_2)^2 p_2^{+2} q^+} - \sum_{q^+ < p_1^+} \ln(\delta e^{\gamma+1} H_{<}) \left[ \frac{2q^{+2} - 2q^+ p_1^+ + 4p_1^{+2}}{p_1^{+3}} \right] \Big\} \\ &\sim -\frac{g^3}{4\pi^2} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) + \sum_{q^+} \left[ \frac{2}{p_{12}^+ - q^+} + \frac{2}{q^+} \right] + \sum_{q^+ \neq p_1^+} \frac{1}{|p_1^+ - q^+|} \right. \\ &+ \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} \ln \frac{p_1^2 (p_1^+ + p_2^+)}{(p_1 + p_2)^2 p_1^+} + \frac{1}{p_{12}^+ - q^+} \ln \frac{p_2^2 (p_1^+ + p_2^+)}{(p_1 + p_2)^2 p_2^+} \right] - \ln \frac{(p_{12}^+)^2}{p_1^+ p_2^+} \\ &+ \int_0^{p_1^+} dq^+ \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+)(p_1^+ - q^+)}{p_1^+ (p_1^+ + p_2^+ - q^+)} \\ &+ \int_{p_1^+}^{p_{12}^+} dq^+ \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+)(q^+ - p_1^+)}{p_2^+ q^+} - \int_0^1 du \ln(\delta e^{\gamma+1} u(1-u)p_1^2) [2u^2 - 2u + 4] \Big\} \end{aligned} \quad (210)$$

$$\begin{aligned} & \text{Surface Terms}^{\wedge \vee \wedge} \\ &= -\frac{g^3}{4\pi^2} \frac{p_2^+}{p_1^+ p_3^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) \right. \end{aligned}$$



$$\begin{aligned}
& + \sum_{q^+ < p_1^+} \left[ \frac{1}{p_{12}^+ - q^+} + \frac{2}{q^+} + \frac{1}{p_1^+ - q^+} \right] + \sum_{q^+ > p_1^+} \left[ \frac{2}{p_{12}^+ - q^+} + \frac{1}{q^+} + \frac{1}{q^+ - p_1^+} \right] \\
& + \sum_{q^+ < p_1^+} \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_1^2(p_1^+ + p_2^+)^2(p_1^+ - q^+)}{(p_1 + p_2)^2 p_1^{+2}(p_1^+ + p_2^+ - q^+)} \\
& + \sum_{q^+ > p_1^+} \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{p_2^2(p_1^+ + p_2^+)^2(q^+ - p_1^+)}{(p_1 + p_2)^2 p_2^{+2} q^+} \\
& - \sum_{q^+ > p_1^+} \ln(\delta e^{\gamma+1} H_>) \left[ \frac{2(p_{12}^+ - q^+)^2 - 2(p_{12}^+ - q^+)p_2^+ + 4p_2^{+2}}{p_2^{+3}} \right] \Big\}
\end{aligned} \tag{211}$$

$$\begin{aligned}
& \sim -\frac{g^3}{4\pi^2} \frac{p_2^+}{p_1^+ p_3^+} K^\wedge \left\{ \mathcal{A}(p_1^2, p_1^+) + \mathcal{A}(p_2^2, p_2^+) + \mathcal{A}(p_3^2, -p_3^+) + \sum_{q^+} \left[ \frac{2}{p_{12}^+ - q^+} + \frac{2}{q^+} \right] + \sum_{q^+ \neq p_1^+} \frac{1}{|p_1^+ - q^+|} \right. \\
& + \sum_{q^+ \neq p_1^+} \left[ \frac{1}{q^+} \ln \frac{p_1^2(p_1^+ + p_2^+)}{(p_1 + p_2)^2 p_1^+} + \frac{1}{p_{12}^+ - q^+} \ln \frac{p_2^2(p_1^+ + p_2^+)}{(p_1 + p_2)^2 p_2^+} \right] - \ln \frac{(p_{12}^+)^2}{p_1^+ p_2^+} \\
& + \int_0^{p_1^+} dq^+ \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+)(p_1^+ - q^+)}{p_1^+(p_1^+ + p_2^+ - q^+)} \\
& \left. + \int_{p_1^+}^{p_{12}^+} dq^+ \left[ \frac{1}{p_{12}^+ - q^+} + \frac{1}{q^+} \right] \ln \frac{(p_1^+ + p_2^+)(q^+ - p_1^+)}{p_2^+ q^+} - \int_0^1 du \ln(\delta e^{\gamma+1} u(1-u)p_2^2) [2u^2 - 2u + 4] \right\}
\end{aligned} \tag{212}$$

Note that the divergent factor multiplying the tree vertex are spin independent.

Next we quote the analogs of Eq. (64) for the other spin configurations:

$$\begin{aligned}
\Gamma_{\Delta-}^{\vee\wedge\wedge} &= \text{Surface Terms} - \frac{g^3}{4\pi^2} \frac{p_1^+}{p_2^+ p_3^+} K^\wedge \sum_{q^+} \int_0^{x_{max}} \frac{dx}{p_{12}^+} \\
& \left\{ - \left[ 2 + \frac{p_2^+}{q^+} + \frac{p_1^+}{(p_{12}^+ - q^+)} \right] I_1 + \frac{1}{q^{+2}} \left[ \frac{q^{+4}}{p_1^{+2}(q^+ - p_1^+)^2} + 1 + \frac{(p_1^+ - q^+)^2}{p_1^{+2}} + \frac{2q^+}{p_{12}^+} \right] I_3 \right. \\
& \left. + \frac{1}{(p_{12}^+ - q^+)^2} \left[ \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + 1 + \frac{2(p_{12}^+ - q^+)}{p_{12}^+} \right] I_2 \right\}
\end{aligned} \tag{213}$$

$$\begin{aligned}
\Gamma_{\Delta-}^{\wedge\vee\wedge} &= \text{Surface Terms} - \frac{g^3}{4\pi^2} \frac{p_2^+}{p_3^+ p_1^+} K^\wedge \sum_{q^+} \int_0^{x_{max}} \frac{dx}{p_{12}^+} \\
& \left\{ - \left[ 2 + \frac{p_2^+}{q^+} + \frac{p_1^+}{(p_{12}^+ - q^+)} \right] I_1 + \frac{1}{q^{+2}} \left[ \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + 1 + \frac{2q^+}{p_{12}^+} \right] I_3 \right. \\
& \left. + \frac{1}{(p_{12}^+ - q^+)^2} \left[ \frac{(p_{12}^+ + q^+)^4}{p_2^{+2}(q^+ - p_1^+)^2} + \frac{(q^+ - p_1^+)^2}{p_2^{+2}} + 1 + \frac{2(p_{12}^+ - q^+)}{p_{12}^+} \right] I_2 \right\}
\end{aligned} \tag{214}$$

The extraction of the  $q^+$  divergences for these spin configurations parallels the discussion in the text. The  $I_1$  term does not contain  $q^+$  divergences as in the text. Again, as in the text, the worst (linear) divergences are for  $q^+ \sim p_1^+$  and arise only from the “First Terms” of  $I_{2,3} = xp_1^+ p_2^+ + \hat{I}_{2,3}$ . This linear divergence is canceled by a term in the corresponding swordfish diagram as in the text:

$$\begin{aligned}
(\text{FirstTerm})^{\vee\wedge\wedge} &= -\frac{g^3}{8\pi^2} \frac{p_1^+}{p_2^+ p_3^+} \frac{K^\wedge}{p_{12}^+} \left\{ \sum_{q^+ < p_1^+} \frac{p_2^+}{p_1^+} \left( \left[ \frac{q^{+4}}{p_1^{+2}(q^+ - p_1^+)^2} + 1 + \frac{(p_1^+ - q^+)^2}{p_1^{+2}} + \frac{2q^+}{p_{12}^+} \right] \right. \right. \\
& \left. \left. + \frac{q^{+2}}{(p_{12}^+ - q^+)^2} \left[ \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + 1 + \frac{2(p_{12}^+ - q^+)}{p_{12}^+} \right] \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{q^+ > p_1^+} \frac{p_1^+}{p_2^+} \left( \frac{(p_{12}^+ - q^+)^2}{q^{+2}} \left[ \frac{q^{+4}}{p_1^{+2}(q^+ - p_1^+)^2} + 1 + \frac{(p_1^+ - q^+)^2}{p_1^{+2}} + \frac{2q^+}{p_{12}^+} \right] \right. \\
& \left. + \left[ \frac{p_2^{+2}}{(q^+ - p_1^+)^2} + 1 + \frac{2(p_{12}^+ - q^+)}{p_{12}^+} \right] \right) \Bigg\} \quad (215)
\end{aligned}$$

$$\begin{aligned}
(\text{FirstTerm})^{\wedge\vee\wedge} = & -\frac{g^3}{8\pi^2} \frac{p_2^+}{p_3^+ p_1^+} \frac{K^\wedge}{p_{12}^+} \left\{ \sum_{q^+ < p_1^+} \frac{p_2^+}{p_1^+} \left( \left[ \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + 1 + \frac{2q^+}{p_{12}^+} \right] \right. \right. \\
& + \frac{q^{+2}}{(p_{12}^+ - q^+)^2} \left[ \frac{(p_{12}^+ + q^+)^4}{p_2^{+2}(q^+ - p_1^+)^2} + \frac{(q^+ - p_1^+)^2}{p_2^{+2}} + 1 + \frac{2(p_{12}^+ - q^+)}{p_{12}^+} \right] \Bigg) \\
& + \sum_{q^+ > p_1^+} \frac{p_1^+}{p_2^+} \left( \frac{(p_{12}^+ - q^+)^2}{q^{+2}} \left[ \frac{p_1^{+2}}{(q^+ - p_1^+)^2} + 1 + \frac{2q^+}{p_{12}^+} \right] \right. \\
& \left. \left. + \left[ \frac{(p_{12}^+ + q^+)^4}{p_2^{+2}(q^+ - p_1^+)^2} + \frac{(q^+ - p_1^+)^2}{p_2^{+2}} + 1 + \frac{2(p_{12}^+ - q^+)}{p_{12}^+} \right] \right) \right\} \quad (216)
\end{aligned}$$

## B.2 The remaining swordfish diagrams

We list here the amplitudes for the other spin configurations of the swordfish diagrams.

$$\begin{aligned}
\Gamma_{SF3}^{\wedge\vee\wedge} = & -\frac{g^3}{8\pi^2} \sum_{q^+} \frac{1}{p_{12}^+} \left\{ \frac{p_{12}^+ - q^+}{q^+} \left[ -3 - \frac{(p_1^+ + q^+)(q^+ - p_1^+ - 2p_2^+)}{(p_1^+ - q^+)^2} \right] \right. \\
& + \left[ \frac{p_{12}^+ - q^+}{q^+} + \frac{q^+}{p_{12}^+ - q^+} \right] \left[ 1 - \frac{(p_{12}^+ - 2q^+)(p_2^+ - p_1^+)}{(p_{12}^+)^2} \right] \Bigg\} (x_3 k_2 + (1 - x_3) k_0)^\wedge \\
= & -\frac{g^3}{8\pi^2} \frac{1}{p_{12}^+} \sum_{q^+} \left\{ \frac{y_3}{1 - y_3} \left[ -3 - \frac{(y_3 + 1 - \eta)(y_3 - 1 - \eta)}{(\eta - 1 + y_3)^2} \right] \right. \\
& \left. + \left[ \frac{1 - y_3}{y_3} + \frac{y_3}{1 - y_3} \right] [1 - (2y_3 - 1)(1 - 2\eta)] \right\} (y_3 k_0 + (1 - y_3) k_2)^\wedge \quad (217)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{SF1}^{\wedge\vee\wedge} = & -\frac{g^3}{8\pi^2} \frac{1}{p_1^+} \sum_{q^+ < p_1^+} \left\{ \frac{1 - x_1}{x_1} \left[ -3 - \frac{(p_{12}^+ + q^+)(q^+ - p_{12}^+)}{(p_{12}^+ - q^+)^2} \right] \right. \\
& + \left[ \frac{1 - x_1}{x_1} + \frac{x_1}{1 - x_1} \right] \left[ 1 - \frac{(2q^+ - p_1^+)(p_1^+ + 2p_2^+)}{p_1^{+2}} \right] \Bigg\} (x_1 k_1 + (1 - x_1) k_0)^\wedge \\
= & -\frac{g^3}{8\pi^2} \frac{1}{p_1^+} \sum_{q^+ < p_1^+} \left\{ \frac{y_1}{1 - y_1} \left[ -3 - \frac{(y_1 + 1 - \eta^{-1})(y_1 - 1 - \eta^{-1})}{(\eta^{-1} - 1 + y_1)^2} \right] \right. \\
& \left. + \left[ \frac{1 - y_1}{y_1} + \frac{y_1}{1 - y_1} \right] [1 - (2y_1 - 1)(1 - 2\eta^{-1})] \right\} (y_1 k_0 + (1 - y_1) k)^\wedge \quad (218)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{SF2}^{\wedge\vee\wedge} = & -\frac{g^3}{8\pi^2} \sum_{q^+ > p_1^+} \frac{p_2^+}{(q^+ - p_1^+)(p_{12}^+ - q^+)} \left[ 1 - \frac{(q^+ - 2p_1^+ - 2p_2^+)(q^+ - 2p_1^+)}{q^{+2}} \right] (x_2 k_1 + (1 - x_2) k_2)^\wedge \\
= & -\frac{g^3}{8\pi^2} \frac{1}{p_2^+} \sum_{q^+ > p_1^+} \frac{1}{x_2(1 - x_2)} \left[ 1 - \frac{(x_2 + (1 - \eta)^{-1})(x_2 + (1 - \eta)^{-1} - 2)}{(x_2 - (1 - \eta)^{-1})^2} \right] (x_2 k_1 + (1 - x_2) k_2)^\wedge \quad (219)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{SF3}^{\vee\wedge\wedge} &= -\frac{g^3}{8\pi^2} \frac{1}{p_{12}^+} \sum_{q^+} \left\{ \frac{q^+}{p_{12}^+ - q^+} \left[ -3 - \frac{(p_1^+ + q^+)(q^+ - p_1^+ - 2p_2^+)}{(p_1^+ - q^+)^2} \right] \right. \\
&\quad \left. + \left[ \frac{p_{12}^+ - q^+}{q^+} + \frac{q^+}{p_{12}^+ - q^+} \right] \left[ 1 - \frac{(p_{12}^+ - 2q^+)(p_2^+ - p_1^+)}{(p_{12}^+)^2} \right] \right\} (x_3 k_2 + (1 - x_3) k_0)^\wedge \\
&= -\frac{g^3}{8\pi^2} \frac{1}{p_{12}^+} \sum_{q^+} \left\{ \frac{x_3}{1 - x_3} \left[ -3 - \frac{(x_3 + \eta)(x_3 + \eta - 2)}{(\eta - x_3)^2} \right] \right. \\
&\quad \left. + \left[ \frac{x_3}{1 - x_3} + \frac{1 - x_3}{x_3} \right] [1 - (2x_3 - 1)(2\eta - 1)] \right\} (x_3 k_2 + (1 - x_3) k_0)^\wedge \tag{220}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{SF1}^{\vee\wedge\wedge} &= -\frac{g^3}{8\pi^2} \frac{1}{p_1^+} \sum_{q^+ < p_1^+} \frac{1}{x_1(1 - x_1)} \left[ 1 - \frac{(p_{12}^+ + q^+)(q^+ - p_{12}^+)}{(p_{12}^+ - q^+)^2} \right] (x_1 k_1 + (1 - x_1) k_0)^\wedge \\
&= -\frac{g^3}{8\pi^2} \frac{1}{p_1^+} \sum_{q^+ < p_1^+} \frac{1}{x_1(1 - x_1)} \left[ 1 - \frac{(x_1 + \eta^{-1})(x_1 + \eta^{-1} - 2)}{(\eta^{-1} - x_1)^2} \right] (x_1 k_1 + (1 - x_1) k_0)^\wedge \tag{221}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{SF2}^{\vee\wedge\wedge} &= -\frac{g^3}{8\pi^2} \sum_{q^+ > p_1^+} \frac{1}{p_2^+} \left\{ \frac{q^+ - p_1^+}{p_{12}^+ - q^+} \left[ -3 - \frac{(q^+ - 2p_1^+ - 2p_2^+)(q^+ - 2p_1^+)}{q^{+2}} \right] \right. \\
&\quad \left. + \left[ \frac{q^+ - p_1^+}{p_{12}^+ - q^+} + \frac{p_{12}^+ - q^+}{q^+ - p_1^+} \right] \left[ 1 - \frac{(2p_{12}^+ - 2q^+)(2p_{12}^+)}{p_2^{+2}} \right] \right\} (x_2 k_1 + (1 - x_2) k_2)^\wedge \\
&= -\frac{g^3}{8\pi^2} \frac{1}{p_2^+} \sum_{q^+ > p_1^+} \left\{ \frac{y_2}{1 - y_2} \left[ -3 - \frac{(y_2 + 1 - (1 - \eta)^{-1})(y_2 - 1 - (1 - \eta)^{-1})}{(y_2 - 1 + (1 - \eta)^{-1})^2} \right] \right. \\
&\quad \left. + \left[ \frac{1 - y_2}{y_2} + \frac{y_2}{1 - y_2} \right] [1 - (2y_2 - 1)(1 - 2(1 - \eta)^{-1})] \right\} (y_2 k_2 + (1 - y_2) k)^\wedge \tag{222}
\end{aligned}$$

In these formulas the  $x_i$ 's are defined as before and the  $y_i \equiv 1 - x_i$ .

Notice the different spin cases involve summands that can be obtained from one another through simple substitution rules. In terms of the rescaled variables  $z = x_i$  or  $y_i$  we only encounter two forms:

$$F_1(z, \xi) = \frac{1}{z(1 - z)} \left[ 1 - \frac{(z + \xi)(z + \xi - 2)}{(\xi - z)^2} \right] (z k_1 + (1 - z) k_2)^\wedge \tag{223}$$

$$\begin{aligned}
F_2(z, \xi) &= \left\{ \frac{z}{1 - z} \left[ -3 - \frac{(z + \xi)(z + \xi - 2)}{(\xi - z)^2} \right] \right. \\
&\quad \left. + \left[ \frac{z}{1 - z} + \frac{1 - z}{z} \right] [1 - (2z - 1)(2\xi - 1)] \right\} (z k_1 + (1 - z) k_2)^\wedge \tag{224}
\end{aligned}$$

where  $k_1, k_2$  are any pair of  $k, k_2, k_0$  and  $\xi$  is one of the variables  $\eta, 1 - \eta, \eta^{-1}, 1 - \eta^{-1}, (1 - \eta)^{-1}, 1 - (1 - \eta)^{-1}$ . It is convenient to decompose each of these expressions into partial fractions:

$$F_1 = \frac{2k_1}{(1 - z)(1 - \xi)} + \frac{2k_2}{z\xi} + 4 \frac{\xi k_1 + (1 - \xi)k_2}{(z - \xi)^2} + 2 \frac{\xi k_1 - (1 - \xi)k_2}{(z - \xi)\xi(1 - \xi)} \tag{225}$$

$$F_2 = \frac{2\xi^2 k_1}{(1 - z)(1 - \xi)} + \frac{2\xi k_2}{z} + 4\xi^2 \frac{\xi k_1 + (1 - \xi)k_2}{(z - \xi)^2} + 2\xi \frac{\xi(5 - 4\xi)k_1 + (1 - \xi)(3 - 4\xi)k_2}{(z - \xi)(1 - \xi)} + \mathcal{P}(z) \tag{226}$$

where  $\mathcal{P}$  is a quadratic polynomial in  $z$ . These decompositions are useful in the evaluations of the singular contributions discussed in the text.

## C Scalar Box

The one-loop scalar box diagram is given by

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q - k_0)^2 (q - k_1)^2 (q - k_2)^2 (q - k_3)^2} \quad (227)$$

It has infra-red divergences when two or more neighboring external lines are on-shell  $(k_{i+1} - k_i)^2 = 0$ , when the propagator connecting two on-shell vertices carries zero momentum, causing the integrand to behave as

$$\frac{1}{(q - k_i)^2} \frac{1}{2(q - k_i) \cdot (k_{i+1} - k_i)} \frac{1}{2(q - k_i) \cdot (k_{i-1} - k_i)} \quad (228)$$

which causes a logarithmic divergence in 4 dimensions. The discretization of  $p^+$  in the context of light-cone quantization doesn't regulate these divergences. To see this, go to the Galilei frame where one of the on-shell particles has  $\mathbf{p} = 0$ . Then the two propagators hooked to this vertex become

$$\frac{1}{\mathbf{q}^2 - 2q^+ q^-} \frac{1}{\mathbf{q}^2 - 2(q^+ - p_1^+) q^-} \quad (229)$$

which show a log divergence near  $(\mathbf{q}, q^-) = 0$ . If the massless particles were instead gluons, this vertex would supply an additional factor of  $\mathbf{q}$  and the divergence would be regulated by discretizing  $p^+$ .

The previous paragraph shows that the box infra-red divergences are identical to certain triangle infra-red divergences. We can use this fact to define a regulated scalar box integral by subtracting from the box integrand a sum of triangle integrands times the IR limit of the fourth (non-diverging propagator). This amounts to supplying a numerator factor

$$\mathcal{N} = 1 - \frac{(q - k_0)^2 + (q - k_2)^2}{(p_1 + p_2)^2} - \frac{(q - k_1)^2 + (q - k_3)^2}{(p_1 + p_4)^2} \quad (230)$$

Introducing Schwinger parameters and completing the square in the exponential leads to

$$\Gamma_{4,S} = \int_0^\infty dT_1 dT_2 dT_3 dT_4 \int \frac{d^4 q}{(2\pi)^4} \mathcal{N} \exp \left\{ -T_{14}(q - K)^2 - \frac{T_1 T_3 (p_1 + p_2)^2 + T_2 T_4 (p_1 + p_4)^2}{T_{14}} \right\} \quad (231)$$

$$\mathcal{N} = 2 \frac{p_{12}^2 + p_{14}^2}{p_{12}^2 p_{14}^2} \left[ \frac{T_1 T_3 p_{12}^2 + T_2 T_4 p_{14}^2}{T_{14}^2} - (q - K)^2 \right] \quad (232)$$

where  $K = (T_2 p_1 + T_3 (p_1 + p_2) - T_4 p_4)/T_{14}$ . Changing variables to  $x_i = T_i/T_{14}$ ,  $T = T_{14}$ ,  $\sum x_i = 1$ , and integrating  $q, T$  yields

$$\begin{aligned} \Gamma_{4,S} &= -\frac{p_{12}^2 + p_{14}^2}{8\pi^2 p_{12}^2 p_{14}^2} \int_{x_2 + x_3 + x_4 \leq 1} dx_2 dx_3 dx_4 \frac{1}{x_1 x_3 (p_1 + p_2)^2 + x_2 x_4 (p_1 + p_4)^2} \\ &= -\frac{p_{12}^2 + p_{14}^2}{8\pi^2 p_{12}^2 p_{14}^2} \int_0^\infty dt \frac{1}{(1+t)(p_{12}^2 t - p_{14}^2)} \ln \frac{p_{12}^2 t}{p_{14}^2} = -\frac{1}{16\pi^2 p_{12}^2 p_{14}^2} \left[ \ln^2 \frac{p_{12}^2}{p_{14}^2} + \pi^2 \right] \end{aligned} \quad (233)$$

## D More Box Integrals

We list here some integrals over Feynman parameters that arise in the box diagrams. We use the shorthand notation  $d^3 x = \prod_{i=1}^4 dx_i \delta(1 - \sum_i x_i)$ .

$$L = \int d^3 x \ln(x_1 x_3 A + x_2 x_4 B) = -\frac{11}{18} + \frac{B \ln B + A \ln A}{6(A+B)} + \frac{AB}{12(A+B)^2} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (234)$$

$$L_1 = \int d^3x \frac{1}{x_1 x_3 A + x_2 x_4 B} = \frac{1}{2(A+B)} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (235)$$

$$L_{1A} = \int d^3x \frac{(x_1, x_3)}{x_1 x_3 A + x_2 x_4 B} = \frac{\ln(A/B)}{2(A+B)} + \frac{B}{4(A+B)^2} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (236)$$

$$L_{1B} = \int d^3x \frac{(x_2, x_4)}{x_1 x_3 A + x_2 x_4 B} = \frac{\ln(B/A)}{2(A+B)} + \frac{A}{4(A+B)^2} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (237)$$

$$L_A = \int d^3x \frac{x_1 x_3}{x_1 x_3 A + x_2 x_4 B} = \frac{1}{6(A+B)} + \frac{B \ln(A/B)}{3(A+B)^2} + \frac{B(B-A)}{12(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (238)$$

$$L_B = \int d^3x \frac{x_2 x_4}{x_1 x_3 A + x_2 x_4 B} = \frac{1}{6(A+B)} + \frac{A \ln(B/A)}{3(A+B)^2} + \frac{A(A-B)}{12(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (239)$$

$$\begin{aligned} L_C &= \int d^3x \frac{(x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_1)}{x_1 x_3 A + x_2 x_4 B} \\ &= -\frac{1}{6(A+B)} + \frac{(A-B) \ln(A/B)}{6(A+B)^2} + \frac{AB}{6(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (240)$$

$$L_{2B} = \int d^3x \frac{(x_2^2, x_4^2)}{x_1 x_3 A + x_2 x_4 B} = \frac{1 - \ln(A/B)}{6(A+B)} - \frac{A \ln(A/B)}{3(A+B)^2} + \frac{A^2}{6(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (241)$$

$$L_{2A} = \int d^3x \frac{(x_1^2, x_3^2)}{x_1 x_3 A + x_2 x_4 B} = \frac{1 - \ln(B/A)}{6(A+B)} - \frac{B \ln(B/A)}{3(A+B)^2} + \frac{B^2}{6(A+B)^3} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \quad (242)$$

$$\begin{aligned} L_{AB} &= \int d^3x \frac{x_1 x_2 x_3 x_4}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{1}{2(A+B)^2} + \frac{(B-A) \ln(A/B)}{2(A+B)^3} + \frac{A^2 + B^2 - 4AB}{12(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (243)$$

$$\begin{aligned} L_{AA} &= \int d^3x \frac{x_1^2 x_3^2}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{A - 2B}{6A(A+B)^2} + \frac{B(5A-B) \ln(A/B)}{6A(A+B)^3} + \frac{B(2B-A)}{6(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (244)$$

$$\begin{aligned} L_{CA} &= \int d^3x \frac{(x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_1) x_1 x_3}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{B - 2A}{6A(A+B)^2} - \frac{(5B-A) \ln(A/B)}{6(A+B)^3} + \frac{B(2A-B)}{6(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (245)$$

$$\begin{aligned} L_{2AB} &= \int d^3x \frac{(x_1^2, x_3^2) x_2 x_4}{(x_1 x_3 A + x_2 x_4 B)^2} \\ &= \frac{A + 4B}{6B(A+B)^2} + \frac{(A-5B) \ln(B/A)}{6(A+B)^3} + \frac{B(B-2A)}{6(A+B)^4} \left( \pi^2 + \ln^2 \frac{A}{B} \right) \end{aligned} \quad (246)$$

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